FIRST CRITICAL FIELD IN THE PINNED THREE DIMENSIONAL GINZBURG–LANDAU MODEL OF SUPERCONDUCTIVITY

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ABSTRACT. We study extreme type-II superconductors described by the three dimensional magnetic Ginzburg–Landau functional incorporating a pinning term $a_\varepsilon(x)$, which we assume to be a bounded measurable function satisfying $b \leq a_\varepsilon(x) \leq 1$ for some constant $b > 0$. A hallmark of such materials is the formation of vortex filaments, which emerge when the applied magnetic field exceeds the first critical field H_{c_1}. In this work, we provide a lower bound for this critical field and provide a characterization of the Meissner solution, that is, the unique vortexless configuration that globally minimizes the energy below H_{c_1}. Moreover, we show that the onset of vorticity is intrinsically linked to a weighted variant of the *isoflux* problem studied in [Rom19a, RSS23]. A crucial role is played by the ε-level tools developed in [Rom19b], which we adapt to the weighted Ginzburg–Landau framework.

1. Introduction

1.1. **The problem and brief state of the art.** Superconductors are specific metals and alloys that, when cooled below a critical temperature (usually extremely low), lose all electrical resistance, enabling currents to flow indefinitely without energy dissipation. This phenomenon was first observed by H. Kamerlingh Onnes in 1911. To describe it, Ginzburg and Landau developed the Ginzburg–Landau model of superconductivity in 1950 [GL50], which has proven highly effective in predicting the behavior of superconductors. This model was later shown to be a limiting case of the Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity (see [BCS57, FHSS12]). The Ginzburg–Landau model is a fundamental contribution to the field of physics, the development of which led to the awarding of several Nobel Prizes.

In this article we will be interested in type-II superconductors, in which regions of normal and superconducting phases can coexist within the material. A hallmark of type-II superconductivity is the formation of quantized vortex filaments — similar in nature to the singularities found in fluid dynamics — that occur when the material is exposed to an external magnetic field. The magnetic field penetrates the superconductor via these vortices, and superconductivity is locally suppressed.

These vortices are not static; they can move due to internal interactions or external forces. Their motion induces an electric field, which leads to energy dissipation and introduces electrical resistance, which in turn effectively degrades the superconducting state. To mitigate this, inhomogeneities may be introduced into the material to act as pinning sites, anchoring

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the vortices and limiting their mobility. For a more detailed discussion of the physics underlying vortex pinning and its implications, see, for instance, [Lik79, CDG96, Tin96, CR97, Pis99] and the references therein.

The Ginzburg–Landau functional with pinning, which models the state of an inhomogeneous superconducting sample in an applied magnetic field, assuming that the temperature is fixed and below the critical one, is

$$GL_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon(x) - |u|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^3} |H - H_{\mathrm{ex}}|^2. \tag{1.1}$$

Here

- Ω is a bounded domain of $\mathbb{R}^3$, that we assume to be simply connected with C^2 boundary.
- $u : \Omega \to \mathbb{C}$ is the *order parameter*. Its squared modulus (the density of Cooper pairs of superconducting electrons in the BCS quantum theory) indicates the local state of the superconductor: where $|u|^2 \approx 1$ the material is in the superconducting phase, where $|u|^2 \approx 0$ in the normal phase.
- $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the electromagnetic vector potential of the induced magnetic field $H = \operatorname{curl} A$.
- ∇_A denotes the covariant gradient $\nabla - iA$.
- $H_{\mathrm{ex}} : \mathbb{R}^3 \to \mathbb{R}^3$ is a given external (or applied) magnetic field. We will assume that $H_{\mathrm{ex}} = h_{\mathrm{ex}} H_{0,\mathrm{ex}}$, where $H_{0,\mathrm{ex}}$ is a fixed vector field and h_{ex} is a real parameter that can be tuned, which represents the intensity of the external field.
- $\varepsilon > 0$ is the inverse of the *Ginzburg–Landau parameter* usually denoted κ, a non-dimensional parameter depending only on the material. We will be interested in the regime of small ε, corresponding to extreme type-II superconductors.
- a_ε is a function that accounts for inhomogeneities in the material. We will assume that $a_\varepsilon \in L^\infty(\Omega)$ and that it takes values in $[b, 1]$, where $b \in (0, 1)$ is a constant independent of ε. The regions where $a_\varepsilon = 1$ correspond to sites without inhomogeneities (we also say that there is no pinning in these regions).

The Ginzburg–Landau model is known to be a $\mathbb{U}(1)$-gauge theory, which means that all the meaningful physical quantities, including the free-energy within Ω

$$F_{\varepsilon,\rho_\varepsilon}(u,A) := \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon(x) - |u|^2)^2 + |\operatorname{curl} A|^2,$$

are invariant under the gauge transformations

$$u \mapsto ue^{i\phi} \quad \text{and} \quad A \mapsto A + \nabla\phi,$$

where ϕ is any sufficiently regular real-valued function. An extremely important quantity in the analysis of the model is the *vorticity*, which is defined, for any sufficiently regular configuration (u, A), as

$$\mu(u,A) = \operatorname{curl}(iu, \nabla_A u) + \operatorname{curl} A,$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$. It corresponds to the gauge-invariant version of twice the *Jacobian* of u. As $\varepsilon \to 0$, the vorticity essentially concentrates in a sum of quantized Dirac masses supported on co-dimension 2 objects. This together with the crucial fact that most of the energy concentrates around these objects is what has essentially allowed mathematicians to analyze the model.

The 2D literature on the Ginzburg–Landau model with pinning is extensive. For a detailed overview of the mathematical aspects of this subject, we refer the interested reader to our recent work [DVR24] and the references therein.

In contrast, the 3D problem remains mostly open. Dos Santos [DS12] considered the problem without external magnetic field and without gauge, that is, when $A = 0$ and $H_{\rm ex} = 0$. By taking a pinning term (independent of ε) which takes some value $b \in (0,1)$ in a non-empty open and strictly convex set compactly contained in Ω, and equal to 1 otherwise, he was able to identify the limiting one dimensional singular set of minimizers of the energy, under the prescription of a Dirichlet boundary condition with non-zero degree. This result extends the classical findings [Riv95, San01] for the homogeneous Ginzburg—Landau model, where the limiting singular set is a mass-minimizing current that corresponds to a minimal connection, a concept introduced in [BCL86]. The pinning modifies the structure of the limiting singular set, which, as identified in [DS12], forms a geodesic link.

Recently, Dos Santos, Rodiac, and Sandier [DSRS23] considered the full Ginzburg–Landau energy with a rapidly oscillating pinning term. They studied both the periodic and random stationary cases, where the oscillations are much faster than ε. Although most of their results concerned the 2D situation, under the assumption that the average (in the periodic case) or the expectation (in the random case) are equal to 1, in 3D they found that in the limit $\varepsilon \to 0$ the limiting functional is the same as that found by [BJOS12] in the homogeneous case, showing that in this special situation one can reduce the study of the problem to the case where the pinning term is replaced by its average or expectation, thus yielding the homogeneous limiting energy.

In the homogeneous case, when $a_\varepsilon \equiv 1$ in Ω, there has been considerable development in the analysis of the model, especially in the last two decades. In particular, Baldo, Jerrard, Orlandi, and Soner [BJOS12, BJOS13] described the asymptotic behavior of the energy functional by deriving a mean-field model as $\varepsilon \to 0$. On the other hand, in [Rom19b], mathematical tools were developed to study the vortex filaments at the ε-level, which has enabled a detailed description of the vortices when the intensity of the applied field is close to the so-called *first critical field* H_{c_1}; see [Rom19a, RSS23, RSS].

Below this value, which is of order $O(|\log\varepsilon|)$, as first established by Abrikosov [Abr57], the superconductor remains fully in its superconducting phase, meaning $|u|$ uniformly close to 1, and the applied field is expelled from the material. However, when the field intensity exceeds H_{c_1}, vortex filaments appear within the material. The order parameter u vanishes in the center of each vortex core, and the applied field penetrates the material through the vortices.

Mathematically, the leading order value of the first critical field for the homogeneous functional in 3D was first derived in [BJOS13]. More precisely, they proved that the vorticity of a minimizing sequence $(u_\varepsilon, A_\varepsilon)$ of GL_ε (when $a_\varepsilon \equiv 1$ in Ω), up to extraction, is such that

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{|\log\varepsilon|} \overset{\varepsilon\to 0}{\longrightarrow} \begin{cases} 0 & \text{if } \lim\limits_{\varepsilon\to 0} \dfrac{h_{\rm ex}}{|\log\varepsilon|} < \dfrac{1}{2\mathrm{R}_0} \\ \text{a non zero limit} & \text{if } \lim\limits_{\varepsilon\to 0} \dfrac{h_{\rm ex}}{|\log\varepsilon|} > \dfrac{1}{2\mathrm{R}_0} \end{cases} \tag{1.2}$$

in a certain weak topology, where the constant R_0 is described later. This result gives H_{c_1} up to an error $o(|\log\varepsilon|)$ as $\varepsilon\to 0$ and agrees with the previous work by Alama, Bronsard, and Montero [ABM06] in the special case where Ω is a ball.

This result was improved in [Rom19a], where it is shown that there exist constants $K_0, K_1 > 0$ such that

$$H_{c_1}^0 - K_0\log|\log\varepsilon| \le H_{c_1} \le H_{c_1}^0 + K_1.$$

More significantly, given a global minimizer $(u_\varepsilon, A_\varepsilon)$ of GL_ε:

- if $h_{\mathrm{ex}} \le H_{c_1}^0 - K_0\log|\log\varepsilon|$, then $\|u_\varepsilon\|_\infty \to 1$ as $\varepsilon\to 0$ and therefore u_ε does not have vortices;
- if $h_{\mathrm{ex}} \ge H_{c_1}^0 + K_1$, then u_ε does have vortices.

In [RSS23], the lower bound was improved, replacing $-K_0\log|\log\varepsilon|$ by a constant $-K_0$ in a special situation. This improvement is heavily based on the boundedness of the vorticity of the minimizers of GL_ε slightly above the first critical field, which is a main result in [RSS23].

To our knowledge, there are no results in the mathematics literature concerning the first critical field for the pinned Ginzburg–Landau functional in 3D, which is the main motivation of our article.

1.2. **Approximation of the Meissner state.** In this paper, we provide a precise approximation of the *Meissner state*, that is, the unique (modulo gauge-invariance) solution of the Ginzburg–Landau equations without vortex filaments, which, in turn, allows us for providing a lower bound for the first critical field, which is of order $O(|\log\varepsilon|)$ as in the homogeneous case. Our approximating configuration is given by

$$(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}}A_\varepsilon^0), \tag{1.3}$$

where

- ρ_ε is the unique positive real-valued minimizer in $H^1(\Omega)$ of the energy functional without magnetic components

$$E_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon - |u|^2)^2. \tag{1.4}$$

  The Euler–Lagrange equation associated to this functional is

$$\begin{cases} -\Delta\rho_\varepsilon &= \dfrac{\rho_\varepsilon}{\varepsilon^2}(a_\varepsilon - \rho_\varepsilon^2) & \text{in } \Omega \\ \dfrac{\partial\rho_\varepsilon}{\partial\nu} &= 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

  Observe that, since a_ε takes values in $[b,1]$, the maximum principle yields the following uniform bounds for ρ_ε.

$$b \le \rho_\varepsilon^2 \le 1. \tag{1.6}$$

- A_ε^0 is the unique minimizer in $[A_{0,\mathrm{ex}} + \dot{H}_\sigma^1(\mathbb{R}^3,\mathbb{R}^3)]$ of the energy functional

$$J_\varepsilon(A) := \frac{1}{2}\int_\Omega \rho_\varepsilon^2|A - \nabla\phi_{A,\rho_\varepsilon^2}|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl}(A - A_{0,\mathrm{ex}})|^2, \tag{1.7}$$

where $\dot H^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)$ is the space of divergence-free vector fields in the homogeneous Sobolev space $\dot H^1(\mathbb{R}^3,\mathbb{R}^3)$ and $\phi_{A,\rho_\varepsilon^2}$ is the unique solution in $H^1(\Omega)$ with zero average, i.e. $\int_\Omega \phi_{A,\rho_\varepsilon^2}=0$, of the elliptic equation

$$\begin{cases} \operatorname{div}\left(\rho_\varepsilon^2(A-\nabla\phi_{A,\rho_\varepsilon^2})\right) = 0 & \text{in } \Omega \\ \rho_\varepsilon^2(A-\nabla\phi_{A,\rho_\varepsilon^2})\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

The structure of the equation implies that

$$A-\nabla\phi_{A,\rho_\varepsilon^2}=\frac{\operatorname{curl} B_A}{\rho_\varepsilon^2} \tag{1.9}$$

for some B_A which satisfies $\operatorname{div} B_A=0$ and $B_A\times\nu=0$. We denote $B_\varepsilon^0=B_{A_\varepsilon^0}$ and $\phi_\varepsilon^0=\phi_{A_\varepsilon^0,\rho_\varepsilon^2}$. Notice that ϕ_ε^0 is phase that appears in (1.3).

As we shall see next, the vector field B_ε^0 is the 3D analog of the function ξ_ε appearing in [DVR24] in the 2D framework. It belongs to the space $C_T^{0,\gamma}(\Omega,\mathbb{R}^3)$ of γ-Hölder-continuous vector fields whose tangential component vanishes on $\partial\Omega$, for any $\gamma\in(0,1]$. The regularity of the vector field B_ε^0 mainly depends on the regularity of $H_{0,\mathrm{ex}}$ in Ω. In particular, assuming henceforth $H_{0,\mathrm{ex}}\in L^3(\Omega,\mathbb{R}^3)$, we have that

$$\|B_\varepsilon^0\|_{C_T^{0,\gamma}(\Omega,\mathbb{R}^3)}\le C \quad \text{for any } \gamma\in(0,1),$$

where the constant $C>0$ does not depend on ε, which is similar to the regularity results for ξ_ε in 2D. This estimate is proved in Proposition 2.2 and is fundamental to our analysis.

It is worth noting that in the 2D setting, we were able to establish that the previous bound holds for $\gamma=1$. We believe this to be the case in 3D as well, although our method of proof does not extend to the case $\gamma=1$ in 3D. Nevertheless, this limitation does not affect our analysis.

A crucial result concerning the special configuration (1.3) is the energy splitting that we present next, which is analogous to [DVR24, Proposition 1.1]. Before doing so, we must define the appropriate space for the minimization of GL_ε. We will assume throughout the article that $H_{\mathrm{ex}}\in L^2_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R}^3)$. Since $\operatorname{div} H_{\mathrm{ex}}=0$ in $\mathbb{R}^3$, in accordance with the nonexistence of magnetic monopoles in Maxwell's theory of electromagnetism, we deduce that there exists a vector potential $A_{\mathrm{ex}}\in H^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R}^3)$ such that

$$\operatorname{curl} A_{\mathrm{ex}}=H_{\mathrm{ex}} \quad \text{in } \mathbb{R}^3.$$

We also define $A_{0,\mathrm{ex}}:=h_{\mathrm{ex}}^{-1}A_{\mathrm{ex}}$. Observe that we can choose A_{ex} in the Coulomb gauge, that is,

$$\begin{cases} \operatorname{div} A_{\mathrm{ex}} = 0 & \text{in } \Omega \\ A_{\mathrm{ex}}\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

This special choice for A_{ex} yields the following useful estimate in Ω, for any $p\in(1,\infty)$,

$$\|A_{\mathrm{ex}}\|_{W^{1,p}(\Omega,\mathbb{R}^3)}\le C\,\|H_{\mathrm{ex}}\|_{L^p(\Omega,\mathbb{R}^3)},$$

where $C=C(\Omega,p)>0$. Similarly,

$$\|A_{0,\mathrm{ex}}\|_{W^{1,p}(\Omega,\mathbb{R}^3)}\le C\,\|H_{0,\mathrm{ex}}\|_{L^p(\Omega,\mathbb{R}^3)}. \tag{1.10}$$

The natural space for the minimization of GL_ε in three dimensions is

$$H^1(\Omega,\mathbb{C})\times[A_{\mathrm{ex}}+H_{\mathrm{curl}}],$$

where

$$H_{\mathrm{curl}} := \{A \in H^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R}^3) \mid \operatorname{curl} A \in L^2(\mathbb{R}^3,\mathbb{R}^3)\}.$$

We are now ready to present our energy splitting.

Proposition 1.1. *Given any configuration* $(\mathbf{u},\mathbf{A}) \in H^1(\Omega,\mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}(\mathbb{R}^3,\mathbb{R}^3)]$*, letting* (u,A) *be defined via the relation* $(\mathbf{u},\mathbf{A}) = (\rho_\varepsilon u e^{ih_{\mathrm{ex}}\phi_\varepsilon}, A + h_{\mathrm{ex}} A^0_\varepsilon)$*, we have*

$$GL_\varepsilon(\mathbf{u},\mathbf{A}) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}}A^0_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u,A) + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A|^2 - h_{\mathrm{ex}}\int_\Omega \mu(u,A)\cdot B^0_\varepsilon + \mathfrak{r}, \tag{1.11}$$

where

$$\mathfrak{r} := \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \frac{|\operatorname{curl} B^0_\varepsilon|^2}{\rho_\varepsilon^2}(|u|^2-1).$$

This splitting was largely driven by a combination of ideas that played a key role in developing the energy splittings presented in [Rom19a, DVR24].

Let us remark that $\mathfrak{r} = \mathfrak{r}(\varepsilon)$ is negligible, in particular, when $h_{\mathrm{ex}} = O(|\log\varepsilon|^m)$ for any $m > 0$. The first term in the RHS of (1.11) captures, with high precision, the minimal energetic cost among vortexless pairs. We have

$$GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi^0_\varepsilon}, h_{\mathrm{ex}}A^0_\varepsilon) = E_\varepsilon(\rho_\varepsilon) + h_{\mathrm{ex}}^2 J_\varepsilon(A^0_\varepsilon),$$

where $E_\varepsilon(\rho_\varepsilon)$ can be interpreted as the energetic cost "enforced" by the potential term in (1.1), whereas the latter term $h_{\mathrm{ex}}^2 J_\varepsilon(A^0_\varepsilon)$ captures the energetic cost associated to the external field.

Let us also observe that, since $\rho_\varepsilon \geq \sqrt{b} > 0$, $\mathbf{u}$ and u have the same vortices and $\mu(\mathbf{u},\mathbf{A}) \approx \mu(u,A)$. For this reason, the second term on the RHS of (1.11) can be interpreted as the energetic cost of the vortex filaments, while the third term represents the magnetic gain associated with them. Hence, the occurrence of vortices strongly depends on the sign of

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\mathrm{ex}}\int_\Omega \mu(u,A)\cdot B^0_\varepsilon. \tag{1.12}$$

1.3. **ε-level estimates.** To analyze the sign of (1.12), the ε-level estimates developed in [Rom19b, Theorem 1.1] become essential. However, their direct application in our context is hindered by the presence of the weight in the free energy $F_{\varepsilon,\rho_\varepsilon}(u,A)$. Our first result overcomes this limitation by generalizing the ε-level tools to the weighted framework.

Theorem 1.1. *Assume* a_ε *is such that:*

(i) *There exist* $\alpha \in (0,1)$*,* $N > 0$*, and* $C_1 > 0$ *(that do not depend on* ε*) such that*

$$\|\rho_\varepsilon\|_{C^{0,\alpha}(\Omega)} \leq C_1|\log\varepsilon|^N.$$

(ii) *There exist* $C_2 > 0$ *and* $\kappa > 0$ *(that do not depend on* ε*) such that* a_ε *is constant in the set* $\{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) \leq C_2|\log\varepsilon|^{-\kappa})\}$.

Then, the following holds. For any $m,n,M > 0$ *there exist* $C,\varepsilon_0 > 0$ *depending only on* $M,m,n,$ *and* $\partial\Omega$ *such that, for any* $\varepsilon < \varepsilon_0$*, if* $(u,A) \in H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^3)$ *satisfies* $F_{\varepsilon,\rho_\varepsilon}(u,A) \leq M|\log\varepsilon|^m$*, then there exists a polyhedral 1-current* ν_ε *such that:*

(1) ν_ε/π *is integer multiplicity,*

(2) $\partial\nu_\varepsilon = 0$ *relative to* Ω*,*

(3) $\mathrm{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} \subset \overline{\Omega}$ *with* $|S_{\nu_\varepsilon}| \leq C|\log\varepsilon|^{-q}$, *where* $q(m,n) := 3(m+n)$,

(4)

$$\int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla_A u|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2 + |\operatorname{curl} A|^2 \geq |\rho_\varepsilon^2 \nu_\varepsilon| \left(|\log\varepsilon| - C\log|\log\varepsilon|\right) - \frac{C}{|\log\varepsilon|^n}, \quad (1.13)$$

where $|\rho_\varepsilon^2\nu_\varepsilon|$ *denotes the mass of the* 1*-current* $\rho_\varepsilon^2\nu_\varepsilon$ *in* Ω[1],

(5) *and for any* $\gamma \in (0,1]$ *there exists* $C_\gamma > 0$ *depending only on* γ *and* $\partial\Omega$ *such that*

$$\|\mu(u,A) - \nu_\varepsilon\|_{(C_T^{0,\gamma}(\Omega,\mathbb{R}^3))^*} \leq C_\gamma \frac{F_{\varepsilon,\rho_\varepsilon}(u,A)+1}{|\log\varepsilon|^{q\gamma}}, \quad (1.14)$$

where $C_T^{0,\gamma}(\Omega,\mathbb{R}^3)$ *denotes the space of* γ*-Hölder continuous vector fields defined in* Ω *whose tangential component vanishes on* $\partial\Omega$.

Remark 1.1. *The hypotheses on* a_ε *are primarily technical and were introduced to facilitate the adaptation of the construction from* [Rom19b] *to the weighted framework. If they do not hold, a different (and more intricate) lower bound is obtained in place of* (1.13)*; see Remark 5.1. The rest of the statement remains valid in this case.*

Using this theorem, we can estimate the difference (1.12) from below, which in turn provides a lower bound for the value of h_{ex} at which the sign of the expression becomes negative, triggering the onset of vortex filament formation. This leads to a weighted version of the *isoflux* problem introduced and analyzed in [Rom19a, RSS23].

1.4. **The weighted isoflux problem and main result.** Given a domain $\Omega \subset \mathbb{R}^3$, we let $\mathcal{N}(\Omega)$ be the space of normal 1-currents supported in $\overline{\Omega}$, with boundary supported on $\partial\Omega$. We denote by $|\cdot|$ the mass of a current. Recall that normal currents are currents with finite mass, whose boundaries have finite mass as well. We also let X denote the class of currents in $\mathcal{N}(\Omega)$ that are simple oriented Lipschitz curves. An element of X must either be a loop contained in $\overline{\Omega}$ or have its two endpoints on $\partial\Omega$.

For any vector field $B \in C_T^{0,1}(\Omega,\mathbb{R}^3)$ and any $\Gamma \in \mathcal{N}(\Omega)$ we denote by $\langle\Gamma, B\rangle$ the value of Γ applied to B, which corresponds to the circulation of the vector field B on Γ when Γ is a curve. If μ is a 2-form then $\langle\mu, B\rangle$ is the scalar product of the 2-form μ and B, seen as a 2-form in this case, in $L^2(\Omega,\Lambda^2(\mathbb{R}^3))$. Let us now recall the definition of the isoflux problem.

Definition 1.1 (Isoflux problem)**.** *The isoflux problem relative to* Ω *and a vector field* $B \in C_T^{0,1}(\Omega,\mathbb{R}^3)$*, is the question of maximizing over* $\mathcal{N}(\Omega)$ *the ratio*

$$\mathrm{R}(\Gamma) := \frac{\langle\Gamma, B\rangle}{|\Gamma|}.$$

Remark 1.2. *In the homogeneous case* $a_\varepsilon \equiv 1$ *in* Ω*, which implies* $\rho_\varepsilon \equiv 1$ *in* Ω*, the constant* R_0 *that appears in* (1.2) *corresponds to the supremum over* $\mathcal{N}(\Omega)$ *of the ratio* R *for the special vector field* B_0 *which appears in the Hodge decomposition of the minimizer* A_0 *of the functional defined in* (1.7) *when* $\rho_\varepsilon \equiv 1$ *in* Ω*, that is,* B_0 *satisfies* (1.9) *with* $\rho_\varepsilon \equiv 1$ *in* Ω.

[1]See Section 1.4 for the notation on currents.

Now, given a smooth function η and $\Gamma \in \mathcal{N}(\Omega)$, we define $\eta\Gamma \in \mathcal{N}(\Omega)$ by duality, that is, for any 1-form ω supported in Ω, we let

$$\langle \eta\Gamma, \omega\rangle := \langle \Gamma, \eta\omega\rangle.$$

Definition 1.2 (Weighted isoflux problem)**.** *The weighted isoflux problem relative to Ω, a smooth weight function η, and a vector field $B \in C_T^{0,1}(\Omega, \mathbb{R}^3)$, is the question of maximizing over $\mathcal{N}(\Omega)$ the ratio*

$$\mathrm{R}_\eta(\Gamma) := \frac{\langle \Gamma, B\rangle}{|\eta\Gamma|}.$$

Remark 1.3. *The existence of maximizers for R_η is ensured by the weak-$\star$ sequential compactness of $\mathcal{N}(\Omega)$. The argument is the same as the one used in the proof of* [RSS23, Theorem 1] *when $\eta \equiv 1$.*

Hereafter we will only be interested in the case $B = B_\varepsilon^0$, where we recall that B_ε^0 is the vector field defined via (1.9) for $A = A_\varepsilon^0$.

By combining the ε-level tools from Theorem 1.1 with the energy splitting (1.11), we deduce that the occurrence of vortex filaments is possible only if $h_{\mathrm{ex}} \geq H_{c_1}^\varepsilon$, where

$$H_{c_1}^\varepsilon := \frac{|\log\varepsilon|}{2\mathrm{R}_\varepsilon} \quad \text{and} \quad \mathrm{R}_\varepsilon := \sup_{\Gamma\in X} \mathrm{R}_{\rho_\varepsilon^2}(\Gamma) = \sup_{\Gamma \in X} \frac{\langle \Gamma, B_\varepsilon^0\rangle}{|\rho_\varepsilon^2\Gamma|}. \tag{1.15}$$

The following is our main result.

Theorem 1.2. *Suppose $\liminf_{\varepsilon\to 0^+} \mathrm{R}_\varepsilon > 0$. If the hypotheses on a_ε in Theorem 1.1 hold, then there exist $\varepsilon_0, K_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$ and $h_{\mathrm{ex}} \leq H_{c_1}^\varepsilon - K_0 \log|\log\varepsilon|$, the global minimizers $(\mathbf{u}, \mathbf{A})$ of GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}(\mathbb{R}^3, \mathbb{R}^3)]$ are such that, letting $(u, A) = \left(\rho_\varepsilon^{-1}\mathbf{u}e^{-ih_{\mathrm{ex}}\phi_\varepsilon}, \mathbf{A} - h_{\mathrm{ex}}A_\varepsilon^0\right)$, as $\varepsilon \to 0$, we have*

(a) $\|\mu(u, A)\|_{\left(C_T^{0,\gamma}(\Omega,\mathbb{R}^3)\right)^*} = o(1)$ *for any $\gamma \in (0, 1]$.*

(b) $GL_\varepsilon(\mathbf{u}, \mathbf{A}) = GL_\varepsilon\left(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}}A_\varepsilon^0\right) + o(1)$.

Moreover, if $(\mathbf{u}, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}(\mathbb{R}^3, \mathbb{R}^3)]$ is a configuration such that $|\mathbf{u}| > c > 0$ for some $c > 0$, $GL_\varepsilon(\mathbf{u}, \mathbf{A}) \leq GL_\varepsilon\left(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}}A_\varepsilon^0\right)$, and $h_{\mathrm{ex}} \leq \varepsilon^{-\alpha}$ for some $\alpha \in \left(0, \frac{1}{2}\right)$, then, as $\varepsilon \to 0$,

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) = GL_\varepsilon\left(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}}A_\varepsilon^0\right) + o(1).$$

The first part of this result characterizes the behavior of global minimizers of GL_ε below $H_{c_1}^\varepsilon$, showing in particular that

$$H_{c_1}^\varepsilon - K_0 \log|\log\varepsilon| \leq H_{c_1}.$$

It essentially states that global minimizers are weakly vortexless, in the sense that their vorticities tend to zero as $\varepsilon \to 0$. Moreover, the minimal energy is, up to a small error, equal to the energy of our approximation of the Meissner state.

In contrast, in the weighted 2D and homogeneous 3D settings, we were able to prove a stronger result. Specifically, we showed that (see [DVR24, Theorem 1.1] and [Rom19a, Theorem 1.2])

$$\|1 - |u|\|_{L^\infty(\Omega,\mathbb{C})} = o(1).$$

This is due to the absence of clearing out in our weighted 3D setting, which constitutes an interesting and challenging open question that may be addressed independently of the present analysis.

The final part of our main result establishes that our approximation of the Meissner state is energetically optimal among vortexless configurations, even when $h_{\rm ex}$ significantly exceeds $O(|\log\varepsilon|)$. In particular, from our energy splitting, we deduce that to obtain a matching upper bound for the first critical field H_{c_1}, it suffices to construct a configuration (u, A) such that (1.12) is negative. One may even take $A = 0$. To achieve this, the main idea is to construct u with a single vortex filament located on an (ε-dependent) 1-current that is nearly optimal for the weighted isoflux problem. This, too, presents a challenging open problem that merits independent investigation.

Finally, we have the following result concerning the first hypothesis of our main result.

Proposition 1.2. *Assume* $\operatorname{curl} H_{0,\rm ex} \not\equiv 0$ *in* Ω. *Then, we have*

$$\liminf_{\varepsilon\to 0^+} \mathrm{R}_\varepsilon > 0.$$

Outline of the paper. The rest of the paper is organized as follows. In Section 2 we provide the tools to construct the approximating Meissner configuration and prove the energy splitting result, Proposition 1.1. Section 3 presents results on lower bounds for solutions of the weighted isoflux problem and establishes Proposition 1.2. In Section 4 we prove our main result, Theorem 1.2. In Section 5 we introduce the appropriate modifications for the proof of the ε-level estimates in the inhomogeneous setting, Theorem 1.1.

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2. Approximating Meissner configuration and Energy splitting

The main goal of this section is to construct the previously mentioned approximation of the Meissner state given by the configuration $\left(\rho_\varepsilon e^{ih_{\rm ex}\phi^0_\varepsilon}, h_{\rm ex}A^0_\varepsilon\right)$ and prove the associated energy splitting result stated in Proposition 1.1. We start by giving an overview on a crucial decoupling result regarding ρ_ε, the unique positive minimizer of the energy functional E_ε defined in (1.4), and some classical results of the L^p-theory of vector fields, which are employed to construct and characterize this special configuration.

2.1. Decoupling of the pinning function.

The following decoupling result establishes a crucial link between inhomogeneous and homogeneous configurations, enabling the generalization of many tools originally developed for the homogeneous case. We observe that E_ε is both coercive and weakly lower semicontinuous in $H^1(\Omega,\mathbb{C})$. This ensures the existence of a minimizer u_ε for E_ε. Moreover, since $E_\varepsilon(|u_\varepsilon|) \le E_\varepsilon(u_\varepsilon)$, we deduce that E_ε admits a positive minimizer. The elliptic structure of minimizers of E_ε given by its Euler–Lagrange equation (1.5) allows us to deduce the following decoupling lemma.

Lemma A. *Let* $u \in H^1(\Omega,\mathbb{C})$ *and* ρ_ε *be a positive minimizer of* E_ε.

(1) *It holds that*

$$E_\varepsilon(\rho_\varepsilon u) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2|\nabla u|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2. \tag{2.1}$$

(2) *Defining the energy with magnetic field by*

$$E_\varepsilon(u,A) := \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon - |u|^2)^2,$$

we have

$$E_\varepsilon(\rho_\varepsilon u, A) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla_A u|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2. \tag{2.2}$$

The result for the energy without magnetic field given in (2.1), was first introduced by Lassoued and Mironescu [LM99] in two dimensions, under a specific pinning function and a prescribed Dirichlet boundary condition. Basically, the same proof applies to any dimension and for any pinning function a_ε, even when modifying the boundary condition for ρ_ε. We refer to [DVR24, Lemma 2.1] for a proof of (2.2) in the 2D setting, which directly extrapolates to the 3D one.

A consequence of (2.1) is the following result.

Lemma B. *There exists a unique positive minimizer ρ_ε for E_ε. Moreover, any other minimizer for E_ε is of the form $\rho_\varepsilon e^{i\theta}$, where θ is a real constant.*

The uniqueness of the positive minimizer was also proven in [LM99], but we refer to [DSRS23, Corollary 2.1] for a proof in our context.

Thus, from now on, ρ_ε denotes the unique positive minimizer of $E_\varepsilon(\cdot)$. This function can be interpreted as a regularized version of $\sqrt{a_\varepsilon}$, which it approximates well. In particular, we have the following result.

Proposition A. *There exist $C, c > 0$ independent of ε such that for any $x \in \Omega, R > 0$, if a_ε is constant in $\Omega \cap B(x,R)$, then*

$$\sup_{\Omega\cap B(x,R)} |\sqrt{a_\varepsilon} - \rho_\varepsilon| \leq Ce^{-c\frac{R}{\varepsilon}}.$$

We refer to [DSM11, Appendix A] for a proof.

2.2. **Notation and some classical results on vector-valued Sobolev spaces.** In this subsection, we state fundamental results on vector fields that will be used to prove the existence and regularity of the approximation of the Meissner configuration. We refer to [KY09, HKS+21] and the references therein for a more detailed analysis on the L^p-theory for vector fields on bounded and exterior domains, respectively. First, let us introduce some important spaces that will appear throughout this section:

- The homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$ with respect to the norm $\|\nabla(\cdot)\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$. A classical result from Ladyzhenskaya [Lad69] shows that in this space the functional

$$\left(\|\mathrm{div}(\cdot)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^3)} + \|\mathrm{curl}(\cdot)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^3)}\right)^{\frac{1}{2}}$$

  defines a norm equivalent to the standard H^1-norm. The space $\dot{H}^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)$ denotes the space of solenoidal (divergence-free) vector fields in $\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$. The inherited norm for this space is just $\|\mathrm{curl}(\cdot)\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$.

- The space $L^2_\sigma(\Omega,\mathbb{R}^3)$ is the space of vector fields in $L^2(\Omega,\mathbb{R}^3)$ with zero divergence in the sense of distributions, that is, $A\in L^2_\sigma(\Omega,\mathbb{R}^3)$ if and only if $A\cdot\nu=0$ on $\partial\Omega$ and for any $\phi\in C_0^\infty(\Omega)$
$$\int_\Omega A\cdot\nabla\phi=0.$$
- The space $H^1_{\sigma,T}(\Omega,\mathbb{R}^3)$ is the space of vector fields $A\in H^1(\Omega,\mathbb{R}^3)$ such that $\operatorname{div}A=0$ and $A\times\nu=0$ on $\partial\Omega$.

The following are classical characterizations of $L^2_\sigma(\Omega,\mathbb{R}^3)$ in the case when Ω does not have holes.

Lemma C (Properties of $L^2_\sigma(\Omega,\mathbb{R}^3)$)**.** *The following holds.*

(1) *The space of smooth compactly supported vector fields with zero divergence is dense in $L^2_\sigma(\Omega,\mathbb{R}^3)$, that is,*
$$\overline{\{\Phi\in C_0^\infty(\Omega,\mathbb{R}^3):\ \operatorname{div}\Phi=0\}}^{\|\cdot\|_{L^2(\Omega,\mathbb{R}^3)}}=L^2_\sigma(\Omega,\mathbb{R}^3). \tag{2.3}$$
(2) *There exists a unique $B_A\in H^1_{\sigma,T}(\Omega,\mathbb{R}^3)$ such that*
$$A=\operatorname{curl}B_A. \tag{2.4}$$
Moreover, we have
$$\|B_A\|_{H^1(\Omega,\mathbb{R}^3)}\le C\,\|A\|_{L^2(\Omega,\mathbb{R}^3)}$$
for some $C=C(\Omega)>0$.

A powerful tool in the analysis of solenoidal and potential fields is the *Helmholtz–Hodge decomposition*, which states that for any bounded or exterior smooth domain $\Omega\subseteq\mathbb{R}^3$
$$L^2(\Omega,\mathbb{R}^3)=L^2_\sigma(\Omega,\mathbb{R}^3)\oplus\nabla H^1(\Omega). \tag{2.5}$$
We will use the following forms of (2.5) given by the geometry of the domain.

Proposition B (Helmholtz–Hodge decomposition)**.** *The following holds.*

(1) *For every $A\in L^2(\Omega,\mathbb{R}^3)$, there exists a unique pair (B_A,ϕ_A), $B_A\in H^1_{\sigma,T}(\Omega,\mathbb{R}^3)$ and $\phi_A\in H^1(\Omega)$ with $\int_\Omega\phi_A=0$ such that*
$$A=\operatorname{curl}B_A+\nabla\phi_A \tag{2.6}$$
This decomposition is continuous in $L^2(\Omega,\mathbb{R}^3)$, that is, there exists $C=C(\Omega)>0$ such that for any $A\in H^1(\Omega,\mathbb{R}^3)$
$$\|\nabla\phi_A\|_{L^2(\Omega)}+\|\operatorname{curl}B_A\|_{L^2(\Omega,\mathbb{R}^3)}\le C\,\|A\|_{L^2(\Omega,\mathbb{R}^3)}\,.$$
(2) *For every $A\in L^2(\mathbb{R}^3,\mathbb{R}^3)$, there exists a unique pair (B_A,ϕ_A), $B_A\in\dot H^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)$ and $\phi_A\in H^1_{\mathrm{loc}}(\mathbb{R}^3)/\mathbb{R}$ with $\nabla\phi_A\in L^2(\mathbb{R}^3,\mathbb{R}^3)$ such that*
$$A=\operatorname{curl}B_A+\nabla\phi_A. \tag{2.7}$$
This decomposition is continuous in $L^2(\mathbb{R}^3,\mathbb{R}^3)$, that is, there exists $C>0$ such that for any $A\in H^1(\mathbb{R}^3,\mathbb{R}^3)$
$$\|\nabla\phi_A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}+\|\operatorname{curl}B_A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}\le C\,\|A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}\,.$$

Remark 2.1. *The decomposition result for bounded domains (2.6) also holds if we replace L^2 by L^p, for any $p\in(1,\infty)$.*

2.3. **Construction of the approximating Meissner configuration.** The Lassoued–Mironescu decoupling (2.2) gives us an intuition on how to identify vortexless configurations. Since $\frac{u}{\rho_\varepsilon}$ behaves like a homogeneous configuration, vortexless configurations should satisfy $\left|\frac{u}{\rho_\varepsilon}\right|^2 \approx 1$, that is, $|u| \approx \rho_\varepsilon$. Thus, a good starting point is to find A in an adequate space such that it minimizes $GL_\varepsilon(\rho_\varepsilon, A)$. However, for reasons that will be explained later (see Remark 2.3), we will introduce a function ϕ_A and minimize $GL_\varepsilon(\rho_\varepsilon e^{i\phi_A}, A) = GL_\varepsilon(\rho_\varepsilon, A - \nabla\phi_A)$ instead. We can use the gauge-invariance property of GL_ε and the Helmholtz–Hodge decomposition (2.7) to assume that $A - A_{0,\mathrm{ex}} \in \dot{H}^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)$ a priori. This particular choice of gauge additionally guarantees coercivity with respect to A.

For convenience's sake, we rescale to minimize $GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}(A - \nabla\phi_A))$. Observe that, from (2.2), we have

$$\begin{aligned} GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}(A - \nabla\phi_A)) &= GL_\varepsilon(\rho_\varepsilon \cdot 1, h_{\mathrm{ex}}(A - \nabla\phi_A)) \\ &= E_\varepsilon(\rho_\varepsilon) + \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega \rho_\varepsilon^2 |A - \nabla\phi_A|^2 + \frac{h_{\mathrm{ex}}^2}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A - H_{0,\mathrm{ex}}|^2. \end{aligned}$$

Therefore, minimizing $GL_\varepsilon(\rho_\varepsilon, h_{\mathrm{ex}}(A - \nabla\phi_A))$ is equivalent to minimizing

$$\frac{1}{2}\int_\Omega \rho_\varepsilon^2 |A - \nabla\phi_A|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl}(A - A_{0,\mathrm{ex}})|^2.$$

Finally, we make the crucial choice (see Remark 2.3) $\phi_A = \phi_{A,\rho_\varepsilon^2}$, the unique solution in $H^1(\Omega)$ with zero-average of (1.8), which leads us to consider the functional J_ε, defined for $A \in [A_{0,\mathrm{ex}} + \dot{H}^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)]$ by

$$J_\varepsilon(A) := \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |A - \nabla\phi_{A,\rho_\varepsilon^2}|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl}(A - A_{0,\mathrm{ex}})|^2.$$

A standard analysis of this functional yields the following result.

Proposition 2.1. *J_ε admists a unique minimizer $A_\varepsilon^0 \in [A_{0,\mathrm{ex}} + \dot{H}^1_\sigma(\mathbb{R}^3, \mathbb{R}^3)]$ which satisfies.*

(a) *$J_\varepsilon(A_\varepsilon^0) \le C \|H_{0,\mathrm{ex}}\|^2_{L^2(\Omega,\mathbb{R}^3)}$ for some $C = C(\Omega) > 0$*

(b) *The Euler–Lagrange equation solved by the minimizer A_ε^0 is given by*

$$\operatorname{curl}(\operatorname{curl} A_\varepsilon^0 - H_{0,\mathrm{ex}}) + \chi_\Omega \operatorname{curl} B_{A_\varepsilon^0,\rho_\varepsilon^2} = 0 \quad \text{in } \mathbb{R}^3, \tag{2.8}$$

where $B_{A_\varepsilon^0,\rho_\varepsilon^2} \in H^1_{\sigma,T}(\Omega, \mathbb{R}^3)$ is the vector field related to A_ε^0 by (1.9) and χ_Ω denotes the indicator function of the set Ω.

(c) *The vector field $B_{A_\varepsilon^0,\rho_\varepsilon^2}$ satisfies the following variational equation*

$$\left\{ \begin{aligned} \int_\Omega \left(\operatorname{curl} \frac{\operatorname{curl} B_{A_\varepsilon^0,\rho_\varepsilon^2}}{\rho_\varepsilon^2} + B_{A_\varepsilon^0,\rho_\varepsilon^2} \right) \cdot V &= \int_\Omega H_{0,\mathrm{ex}} \cdot V && \text{in } \Omega \\ B_{A_\varepsilon^0,\rho_\varepsilon^2} \times \nu &= 0 && \text{on } \Omega, \end{aligned} \right. \tag{2.9}$$

for all $V \in L^2_\sigma(\Omega, \mathbb{R}^3)$.

Before proving this proposition, let us introduce some notation regarding the minimizer A_ε^0 that will be used throughout the rest of the paper. We denote $H_\varepsilon^0 := \operatorname{curl} A_\varepsilon^0$. Also, from (1.8) and (1.9) for $A = A_\varepsilon^0$, we get a function $\phi_{A_\varepsilon^0,\rho_\varepsilon^2}$ and a vector field $B_{A_\varepsilon^0,\rho_\varepsilon^2}$, which we will denote respectively by ϕ_ε^0 and B_ε^0.

Proof. We split the proof into two steps.

Step 1. Existence and uniqueness. Proof of item (a). Since the standard H^1-norm in $\dot{H}^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)$ is equivalent to $\|\operatorname{curl} A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$, the coercivity of J_ε in $[A_{0,ex}+\dot{H}^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)]$ follows directly. On the other hand, by the linearity of equation (1.8), we deduce that the map $A\to\nabla\phi_{A,\rho_\varepsilon^2}$ is linear. Furthermore, by using $\phi_{A,\rho_\varepsilon^2}$ as a test function in (1.8), we have

$$\int_\Omega \rho_\varepsilon^2|\nabla\phi_{A,\rho_\varepsilon^2}|^2=\int_\Omega \rho_\varepsilon^2 A\cdot\nabla\phi_{A,\rho_\varepsilon^2}.$$

By the Cauchy-Schwarz inequality and (1.6), we conclude that the map $A\to\nabla\phi_{A,\rho_\varepsilon^2}$ is a bounded linear operator from $L^2(\Omega,\mathbb{R}^3)$ to $L^2(\Omega,\mathbb{R}^3)$. Therefore, J_ε is both (strongly) continuous and strictly convex, which implies that it is also weakly lower semicontinuous. It thus follows from the direct method of the calculus of variations that J_ε admits a unique minimizer in $[A_{0,\mathrm{ex}}+\dot{H}^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)]$ denoted by A_ε^0. Finally, we see that item (a) is satisfied by noticing that $A_{0,\mathrm{ex}}$ is admissible for J_ε, since, by recalling the boundedness of the operator $A\to\nabla\phi_{A,\rho_\varepsilon^2}$, we have

$$\begin{aligned}J_\varepsilon(A_\varepsilon^0)\le J_\varepsilon(A_{0,\mathrm{ex}})&=\frac12\int_\Omega\rho_\varepsilon^2|A_{0,\mathrm{ex}}-\nabla\phi_{A_{0,\mathrm{ex}},\rho_\varepsilon^2}|^2\\&\overset{(1.6)}{\le} C\|A_{0,\mathrm{ex}}\|^2_{L^2(\Omega,\mathbb{R}^3)}\\&\overset{(1.10)}{\le} C\|H_{0,\mathrm{ex}}\|^2_{L^2(\Omega,\mathbb{R}^3)}.\end{aligned}$$

Step 2. Euler–Lagrange equation for A_ε^0. Proof of items (b) and (c). From minimality of A_ε^0, we deduce that it satisfies, for all $A\in\dot{H}^1_\sigma(\mathbb{R}^3,\mathbb{R}^3)$,

$$\int_\Omega\rho_\varepsilon^2(A_\varepsilon^0-\nabla\phi_\varepsilon^0)\cdot(A-\nabla\phi_{A,\rho_\varepsilon^2})+\int_{\mathbb{R}^3}(H_\varepsilon^0-H_{0,\mathrm{ex}})\cdot\operatorname{curl}A=0. \tag{2.10}$$

Since $\rho_\varepsilon^2(A_\varepsilon^0-\nabla\phi_\varepsilon^0)\in L^2_\sigma(\Omega,\mathbb{R}^3)$, it follows from (2.4) that we can write $\rho_\varepsilon^2(A_\varepsilon^0-\nabla\phi_\varepsilon^0)$ as $\operatorname{curl}B_\varepsilon^0$, where $B_\varepsilon^0\in H^1_{\sigma,T}(\Omega,\mathbb{R}^3)$. It is worth noting that B_ε^0 satisfies

$$\begin{cases}\operatorname{div}B_\varepsilon^0 &= 0 & \text{in }\Omega\\ \operatorname{curl}B_\varepsilon^0 &= \rho_\varepsilon^2(A_\varepsilon^0-\nabla\phi_\varepsilon^0) & \text{in }\Omega\\ B_\varepsilon^0\times\nu &= 0 & \text{on }\partial\Omega\\ \operatorname{curl}B_\varepsilon^0\cdot\nu &= 0 & \text{on }\partial\Omega.\end{cases}\tag{2.11}$$

Additionally, integrating by parts we deduce the orthogonality relation, for any $\phi\in H^1(\Omega)$,

$$\int_\Omega\rho_\varepsilon^2(A_\varepsilon^0-\nabla\phi_\varepsilon^0)\cdot\nabla\phi\overset{(2.11)}{=}\int_\Omega\operatorname{curl}B_\varepsilon^0\cdot\nabla\phi=\int_\Omega B_\varepsilon^0\cdot\operatorname{curl}\nabla\phi-\int_{\partial\Omega}\nabla\phi\cdot(B_\varepsilon^0\times\nu)\overset{(2.11)}{=}0. \tag{2.12}$$

Using (2.12) with $\phi=\phi_{A,\rho_\varepsilon^2}$ and inserting it into equation (2.10), we obtain

$$\int_\Omega\operatorname{curl}B_\varepsilon^0\cdot A+\int_{\mathbb{R}^3}(H_\varepsilon^0-H_{0,\mathrm{ex}})\cdot\operatorname{curl}A=0. \tag{2.13}$$

By the Helmholtz–Hodge decomposition (2.7), we conclude that (2.13) holds for any $A\in\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$. In particular, for any $\Phi\in C_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}(\chi_\Omega\operatorname{curl}B_\varepsilon^0+\operatorname{curl}(H_\varepsilon^0-H_{0,\mathrm{ex}}))\cdot\Phi=0.$$

This yields that the Euler–Lagrange equation for J_ε is (2.8), concluding the proof of item (b).

Item (c) follows from using vector fields supported in Ω as test vector fields in (2.13). If $\Phi \in C_0^\infty(\Omega,\mathbb{R}^3)$, it follows from equation (2.13) and integration by parts that

$$\int_\Omega (B_\varepsilon^0 + H_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl}\Phi = 0.$$

Now let $\Psi \in C_0^\infty(\Omega,\mathbb{R}^3)$ such that $\operatorname{div}\Psi = 0$ in Ω. Since, by (2.4), we can write Ψ as $\operatorname{curl}\Phi$ in Ω with $\Phi\times\nu = 0$ on $\partial\Omega$, we have

$$\int_\Omega (B_\varepsilon^0 + H_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot \Psi = 0.$$

Using the density property (2.3) of $L^2_\sigma(\Omega,\mathbb{R}^3)$, we have that for any $A\in L^2_\sigma(\Omega,\mathbb{R}^3)$,

$$\int_\Omega (B_\varepsilon^0 + H_\varepsilon^0 - H_{0,\mathrm{ex}}) \cdot A \overset{(2.11)}{=} \int_\Omega \left(B_\varepsilon^0 + \operatorname{curl}\frac{\operatorname{curl} B_\varepsilon^0}{\rho_\varepsilon^2} - H_{0,\mathrm{ex}}\right)\cdot A = 0.$$

This concludes the proof of item (c).

□

2.4. **Regularity of the Meissner configuration.** Although we can deduce differenciability for the minimizer using standard regularity results, we are specifically looking for uniform bounds that do not depend on ε. For example, we can deduce from item (a) of Proposition 2.1, using (1.6), that

$$\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)} + \left\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\right\|_{H^1(\mathbb{R}^3,\mathbb{R}^3)} \le C\left\|A_{0,\mathrm{ex}}\right\|_{L^2(\Omega,\mathbb{R}^3)} \overset{(1.10)}{\le} C\left\|H_{0,\mathrm{ex}}\right\|_{L^2(\Omega,\mathbb{R}^3)}, \quad (2.14)$$

where $C = C(\Omega, b) > 0$. Moreover, based on the weak Euler–Lagrange equation (2.13), we obtain the following result.

Proposition 2.2. *Let A_ε^0 be the minimizer of J_ε given by Proposition 2.1, and B_ε^0 be given by (1.9). For any $q\in[1,\infty)$, $B_\varepsilon^0 \in W^{1,q}(\Omega,\mathbb{R}^3)$ with*

$$\left\|B_\varepsilon^0\right\|_{W^{1,q}(\Omega,\mathbb{R}^3)} \le C\left\|H_{0,\mathrm{ex}}\right\|_{L^3(\Omega,\mathbb{R}^3)}, \quad (2.15)$$

where $C = C(\Omega,b,q) > 0$. Furthermore, $B_\varepsilon^0\in C_T^{0,\gamma}(\Omega,\mathbb{R}^3)$ for all $\gamma\in(0,1)$, with

$$\left\|B_\varepsilon^0\right\|_{C^{0,\gamma}(\Omega,\mathbb{R}^3)} \le C\left\|H_{0,\mathrm{ex}}\right\|_{L^3(\Omega,\mathbb{R}^3)}, \quad (2.16)$$

where $C = C(\Omega,b,\gamma) > 0$.

Proof. By using interior elliptic regularity on equation (2.8), we have, for an open ball $B\subset\mathbb{R}^3$ that compactly contains Ω, say $B = B(x_0, \operatorname{diam}(\Omega)+1)$ for $x_0\in\Omega$, that there exists $C = C(\Omega) > 0$ such that

$$\left\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\right\|_{H^2(B,\mathbb{R}^3)} \le C\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)}.$$

In particular, we have

$$\left\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\right\|_{H^2(\Omega,\mathbb{R}^3)} \le C\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)}, \quad (2.17)$$

where $C = C(\Omega) > 0$. On the other hand, for $q \in (1, \infty)$,

$$\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} \overset{(2.11)}{=} \left\|\rho_\varepsilon^2(A_\varepsilon^0 - \nabla\phi_\varepsilon^0)\right\|_{L^q(\Omega,\mathbb{R}^3)} \overset{(1.6)}{\leq} C\left(\left\|A_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} + \left\|\nabla\phi_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)}\right). \tag{2.18}$$

We aim to bound $\|\nabla\phi_\varepsilon^0\|_{L^q(\Omega,\mathbb{R}^3)}$ in terms of A_ε^0, by exploiting the elliptic structure of equation (1.8). Using the dual characterization of the L^q-norm, we have, for any $q \in (1, \infty)$, that

$$\left\|\nabla\phi_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} = \sup_{V \in L^r(\Omega,\mathbb{R}^3)} \frac{1}{\|V\|_{L^r(\Omega,\mathbb{R}^3)}} \int_\Omega V \cdot \nabla\phi_\varepsilon^0,$$

where r is such that $\frac{1}{q} + \frac{1}{r} = 1$. Using the Helmholtz–Hodge decomposition (2.6) (see Remark 2.1), we can write V as $\operatorname{curl} B_V + \nabla\varphi_V$, where $\operatorname{curl} B_V \in L^q_\sigma(\Omega, \mathbb{R}^3)$, that is, it satisfies for any $\zeta \in W^{1,r}(\Omega)$

$$\int_\Omega \operatorname{curl} B_V \cdot \nabla\zeta = 0$$

and

$$\|\operatorname{curl} B_V\|_{L^q(\Omega,\mathbb{R}^3)} + \|\nabla\varphi_V\|_{L^q(\Omega,\mathbb{R}^3)} \leq C\,\|V\|_{L^q(\Omega,\mathbb{R}^3)}$$

for $C = C(\Omega, q) > 0$. We can therefore deduce that

$$\int_\Omega \operatorname{curl} B_V \cdot \nabla\phi_\varepsilon^0 = 0$$

and the continuity of the decomposition yields a constant $C = C(\Omega, q) > 0$, independent of ε, such that

$$\left\|\nabla\phi_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} \leq C \sup_{\nabla\varphi \in L^r(\Omega,\mathbb{R}^3)} \frac{1}{\|\nabla\varphi\|_{L^r(\Omega,\mathbb{R}^3)}} \int_\Omega \nabla\varphi \cdot \nabla\phi_\varepsilon^0. \tag{2.19}$$

Note that both the continuity and the orthogonality of the components allow us to remove the dependence on V for $\nabla\varphi$.

Now, using (1.6), from (2.19) we deduce that

$$\begin{aligned}
\left\|\nabla\phi_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} &\leq C \sup_{\nabla\varphi \in L^r(\Omega,\mathbb{R}^3)} \frac{1}{\|\nabla\varphi\|_{L^r(\Omega,\mathbb{R}^3)}} \int_\Omega \nabla\varphi \cdot \rho_\varepsilon^2 \nabla\phi_\varepsilon^0 \\
&\overset{(1.8)}{=} C \sup_{\nabla\varphi \in L^r(\Omega,\mathbb{R}^3)} \frac{1}{\|\nabla\varphi\|_{L^r(\Omega,\mathbb{R}^3)}} \int_\Omega \nabla\varphi \cdot \rho_\varepsilon^2 A_\varepsilon^0 \\
&\leq C\left\|\rho_\varepsilon^2 A_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} \\
&\overset{(1.6)}{\leq} C\left\|A_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)}.
\end{aligned}$$

Inserting this into (2.18), yields

$$\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} \leq C\left\|A_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)}.$$

Thus, by the continuous embeddings $H^2(\Omega,\mathbb{R}^3)\hookrightarrow L^q(\Omega,\mathbb{R}^3)$ and $W^{1,3}(\Omega,\mathbb{R}^3)\hookrightarrow L^q(\Omega,\mathbb{R}^3)$, we have

$$\begin{aligned}\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} &\le C\left(\left\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\right\|_{L^q(\Omega,\mathbb{R}^3)} + \left\|A_{0,\mathrm{ex}}\right\|_{L^q(\Omega,\mathbb{R}^3)}\right)\\ &\le C\left(\left\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\right\|_{H^2(\Omega,\mathbb{R}^3)} + \left\|A_{0,\mathrm{ex}}\right\|_{W^{1,3}(\Omega,\mathbb{R}^3)}\right)\\ &\overset{(1.10)\&(2.17)}{\le} C\left(\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)} + \left\|H_{0,\mathrm{ex}}\right\|_{L^3(\Omega,\mathbb{R}^3)}\right)\\ &\overset{(2.14)}{\le} C\left(\left\|H_{0,\mathrm{ex}}\right\|_{L^2(\Omega,\mathbb{R}^3)} + \left\|H_{0,\mathrm{ex}}\right\|_{L^3(\Omega,\mathbb{R}^3)}\right)\\ &\le C\left\|H_{0,\mathrm{ex}}\right\|_{L^3(\Omega,\mathbb{R}^3)}.\end{aligned}$$

Finally, due to the boundary condition satisfied by B_ε^0, we can apply a Friedrichs–Poincaré type inequality (see for instance [AS13, Corollary 3.2]), to obtain

$$\left\|B_\varepsilon^0\right\|_{W^{1,q}(\Omega,\mathbb{R}^3)} \le C\left(\left\|\operatorname{div} B_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)} + \left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^q(\Omega,\mathbb{R}^3)}\right) \overset{(2.11)\&(2.14)}{\le} C\left\|H_{0,\mathrm{ex}}\right\|_{L^3(\Omega,\mathbb{R}^3)},$$

where $C=C(\Omega,b,q)>0$ does not depend on ε. By Sobolev embedding for any $q>3$, we have

$$\left\|B_\varepsilon^0\right\|_{C^{0,\gamma}(\Omega,\mathbb{R}^3)} \le C\left\|B_\varepsilon^0\right\|_{W^{1,q}(\Omega,\mathbb{R}^3)} \le C\left\|H_{0,\mathrm{ex}}\right\|_{L^3(\Omega,\mathbb{R}^3)},$$

where $\gamma = 1-\frac{3}{q}\in(0,1)$. □

Remark 2.2. *Standard elliptic regularity results for the equations satisfied by A_ε^0 and ϕ_ε^0 yield that $B_\varepsilon^0\in C_T^{0,1}(\Omega,\mathbb{R}^3)$. Nevertheless, our argument for obtaining a uniform bound in ε fails when $q=\infty$ or $\gamma=1$. The difficulty stems from the lack of a continuous Helmholtz–Hodge decomposition (2.6) in $L^1(\Omega,\mathbb{R}^3)$, which would be required to estimate the L^∞-norm of $\nabla\phi_\varepsilon^0$ in a manner that leverages the weak formulation of equation (1.8).*

2.5. **Proof of the Proposition 1.1.** We finish this section with a proof of the energy splitting result (1.11).

Proof. We start by using (2.2) to decouple the energy in Ω. We have

$$GL_\varepsilon(\mathbf{u},\mathbf{A}) = E_\varepsilon(\rho_\varepsilon) + \frac12\int_\Omega\left(\rho_\varepsilon^2|\nabla_{\mathbf{A}}(ue^{ih_{\mathrm{ex}}\phi_\varepsilon^0})|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2\right) + \frac12\int_{\mathbb{R}^3}|\operatorname{curl} A + h_{\mathrm{ex}}\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2. \quad (2.20)$$

First, by expanding the square in the second term in the RHS of (2.20), we have

$$\begin{aligned}\int_\Omega \rho_\varepsilon^2|\nabla_{\mathbf{A}}(ue^{ih_{\mathrm{ex}}\phi_\varepsilon^0})|^2 &= \int_\Omega \rho_\varepsilon^2|\nabla_A u + ih_{\mathrm{ex}}(\nabla\phi_\varepsilon^0 - A_\varepsilon^0)u|^2\\ &\overset{(2.11)}{=} \int_\Omega \rho_\varepsilon^2|\nabla_A u|^2 + h_{\mathrm{ex}}^2\rho_\varepsilon^2|\nabla\phi_\varepsilon^0 - A_\varepsilon^0|^2|u|^2 - 2h_{\mathrm{ex}}\langle iu,\nabla_A u\rangle\cdot\operatorname{curl} B_\varepsilon^0.\end{aligned}$$

On the other hand, by expanding the square in the rightmost term of (2.20), we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\operatorname{curl} A + h_{\mathrm{ex}} \operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2 &= \int_{\mathbb{R}^3} |\operatorname{curl} A|^2 + h_{\mathrm{ex}}^2 |\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2 \\ &\qquad + 2h_{\mathrm{ex}} \int_{\mathbb{R}^3} \operatorname{curl} A \cdot \operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}}) \\ &\overset{(2.13)}{=} \int_{\mathbb{R}^3} |\operatorname{curl} A|^2 + h_{\mathrm{ex}}^2 |\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2 \\ &\qquad - 2h_{\mathrm{ex}} \int_\Omega \operatorname{curl} B_\varepsilon^0 \cdot A.\end{aligned}$$

Since $B_\varepsilon^0 \times \nu = 0$ on $\partial\Omega$, by integration by parts, we have

$$\int_\Omega (\langle iu, \nabla_A u\rangle + A) \cdot \operatorname{curl} B_\varepsilon^0 = \int_\Omega \mu(u, A) \cdot B_\varepsilon^0. \tag{2.21}$$

Combining our previous computations and inserting into (2.20), we then deduce that

$$\begin{aligned}GL_\varepsilon(\mathbf{u}, \mathbf{A}) &= E_\varepsilon(\rho_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u, A) + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A|^2 - h_{\mathrm{ex}} \int_\Omega \mu(u, A) \cdot B_\varepsilon^0 \\ &\qquad + \frac{h_{\mathrm{ex}}^2}{2} \int_\Omega \left(\rho_\varepsilon^2 |\nabla \phi_\varepsilon^0 - A_\varepsilon^0|^2 |u|^2 + \int_{\mathbb{R}^3} |\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex}})|^2 \right).\end{aligned} \tag{2.22}$$

Finally, recalling that

$$GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}} A_\varepsilon^0) = E_\varepsilon(\rho_\varepsilon) + \frac{h_{\mathrm{ex}}^2}{2} \left(\int_\Omega \rho_\varepsilon^2 |\nabla \phi_\varepsilon^0 - A_\varepsilon^0|^2 + \int_{\mathbb{R}^3} |\operatorname{curl}(A_\varepsilon^0 - A_{0,\mathrm{ex})}|^2 \right),$$

we conclude the proof by writing $A_\varepsilon^0 - \nabla\phi_\varepsilon$ as $\dfrac{\operatorname{curl} B_\varepsilon^0}{\rho_\varepsilon^2}$ and $|u|^2$ as $1 + (|u|^2 - 1)$ in (2.22), which yields

$$\begin{aligned}GL_\varepsilon(\mathbf{u}, \mathbf{A}) &= GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}} A_\varepsilon^0) + F_{\varepsilon,\rho_\varepsilon}(u, A) + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A|^2 \\ &\qquad - h_{\mathrm{ex}} \int_\Omega \mu(u, A) \cdot B_\varepsilon^0 + \frac{h_{\mathrm{ex}}^2}{2} \int_\Omega \frac{|\operatorname{curl} B_\varepsilon^0|^2}{\rho_\varepsilon^2} (|u|^2 - 1).\end{aligned}$$

□

Remark 2.3. *Recalling* (1.8)*, we emphasize that the specific choice of* $\phi^0_{A,\rho_\varepsilon^2}$ *is what guarantees the validity of* (1.9)*. This is a key ingredient when integrating parts in* (2.21)*, as it enables the vorticity term to emerge in our energy decomposition. The function* $\phi^0_{A,\rho_\varepsilon^2}$ *thus serves the same purpose as the function* ϕ_0 *in the splitting result of* [Rom19a, Proposition 3.1]*, which pertains to the homogeneous case.*

3. Weighted isoflux problem

This section is devoted to the proof of Proposition 1.2. To establish the result, we begin by demonstrating a key property of the weighted isoflux problem (see Definition 1.2), which serves as a foundational component of our argument.

First, let us introduce a family of currents induced by differential forms. Any 2-form α supported in $\overline{\Omega}$ induces a 1-current $[\alpha]$, defined by its action on 1-forms ω as

$$\langle [\alpha], \omega \rangle := \int_\Omega \alpha \wedge \omega.$$

For $[\alpha]$ to be an admissible current for the isoflux problem—that is, $[\alpha] \in \mathcal{N}(\Omega)$—it is necessary and sufficient that α be a closed 2-form, which is formalized in the following proposition.

Proposition 3.1. *Let α is a smooth, continuous up to the boundary and supported in $\overline{\Omega}$. We have $[\alpha] \in \mathcal{N}(\Omega)$ if and only if $d\alpha = 0$. Furthermore, its mass is equal to*

$$|[\alpha]| = \|\alpha\|_{L^1(\Omega)}, \tag{3.1}$$

and, in the case $[\alpha] \in \mathcal{N}(\Omega)$, its boundary $\partial[\alpha]$ is the 0-current, defined by its action on 0-forms f as

$$\langle \partial[\alpha], f \rangle = \int_{\partial\Omega} f\alpha.$$

Proof. We begin by proving that $[\alpha]$ has finite mass. Let $\mathcal{D}^1(\Omega)$ be the space of smooth, compactly supported 1-forms in Ω. By the dual characterization of L^p norms, we have

$$\begin{aligned} |[\alpha]| &= \sup_{\omega \in \mathcal{D}^1(\Omega), \|\omega\|_{L^\infty(\Omega)}=1} \langle [\alpha], \omega \rangle \\ &= \sup_{\omega \in \mathcal{D}^1(\Omega), \|\omega\|_{L^\infty(\Omega)}=1} \int_\Omega \alpha \wedge \omega \\ &= \|\alpha\|_{L^1(\Omega)} < +\infty. \end{aligned}$$

Note that α is integrable, as it is continuous in $\overline{\Omega}$.

We now consider $\partial[\alpha]$. Let f be a smooth 0-form, that is, f is a smooth function. In this context, the wedge product reduces to ordinary multiplication. Therefore, by Stokes' theorem, we have

$$\langle \partial[\alpha], f \rangle = \langle [\alpha], df \rangle = \int_\Omega \alpha \wedge df = \int_\Omega d(f\alpha) - f d\alpha = \int_{\partial\Omega} f\alpha - \int_\Omega f d\alpha.$$

If $d\alpha = 0$, the current reduces to

$$\langle \partial[\alpha], f \rangle = \int_{\partial\Omega} f\alpha.$$

This means that $\partial[\alpha]$ is supported on $\partial\Omega$. In addition, by arguing exactly as when computing $|[\alpha]|$, we find

$$|\partial[\alpha]| = \|\alpha\|_{L^1(\partial\Omega)} < +\infty.$$

We have thus proven that $[\alpha] \in \mathcal{N}(\Omega)$. On the other hand, if $[\alpha] \in \mathcal{N}(\Omega)$, we then have that $\partial[\alpha]$ is supported in $\partial\Omega$. Therefore, for any $\phi \in C_0^\infty(\Omega)$, we have

$$\langle \partial[\alpha], \phi \rangle = -\int_\Omega \phi d\alpha = 0.$$

This implies that $d\alpha = 0$. $\square$

We can now state a lower bound for $\max_{\Gamma\in\mathcal{N}(\Omega)} \mathrm{R}_\eta(\Gamma)$ in the special case when B is a solenoidal vector field (recall Definition 1.2).

Proposition 3.2. *Suppose* $\operatorname{div} B = 0$ *in* Ω. *Then, we have*

$$\max_{\Gamma\in\mathcal{N}(\Omega)} \mathrm{R}_\eta(\Gamma) \geq \frac{\|B\|_{L^2(\Omega,\mathbb{R}^3)}}{\|\eta\|_{L^2(\Omega)}}. \tag{3.2}$$

Proof. Writing $B = B_x dx + B_y dy + B_z dz$, we have that $\star B$ is the 2-form $B_x dy\wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$, where $\star$ denotes the Hodge star operator. Moreover, $\star B$ is closed since $d(\star B) = (\operatorname{div} B) dx \wedge dy \wedge dz = 0$. Thus, by Proposition 3.1, $[\star B] \in \mathcal{N}(\Omega)$ and $|[\star B]| = \|B\|_{L^1(\Omega,\mathbb{R}^3)}$. Moreover, it is not hard to see that $\eta[\star B]$ coincides with the current induced by the product $[\eta \star B]$. This implies by (3.1) that $|\eta[\star B]| = \|\eta B\|_{L^1(\Omega,\mathbb{R}^3)}$. Additionally, we have $\star B \wedge B = (|B_x|^2 + |B_y|^2 + |B_z|^2) dx \wedge dy \wedge dz$. By using $[\star B]$ as a competitor for the weighted isoflux problem, we have

$$\begin{aligned}\max_{\Gamma\in\mathcal{N}} \mathrm{R}_\eta(\Gamma) \geq \mathrm{R}_\eta([\star B]) = \frac{\langle [\star B]\,, B\rangle}{|\eta[\star B]|} \overset{(3.1)}{=} \frac{1}{\|\eta B\|_{L^1(\Omega,\mathbb{R}^3)}} \int_\Omega \star B \wedge B &\geq \frac{\|B\|^2_{L^2(\Omega,\mathbb{R}^3)}}{\|\eta\|_{L^2(\Omega)}\|B\|_{L^2(\Omega,\mathbb{R}^3)}} \\ &= \frac{\|B\|_{L^2(\Omega,\mathbb{R}^3)}}{\|\eta\|_{L^2(\Omega)}}.\end{aligned}$$

□

3.1. **Proof of Proposition 1.2.** We have all the ingredients to provide a proof for Proposition 1.2, which we recall applies to the special case $B = B_\varepsilon^0$ and $\eta = \rho_\varepsilon^2$; see (1.15).

Proof of Proposition 1.2. Assume $\liminf_{\varepsilon\to 0^+} \mathrm{R}_\varepsilon = 0$. Since $\operatorname{div} B_\varepsilon^0 = 0$ in Ω and (1.6), from (3.2) it follows that, by passing to a subsequence if necessary, $B_\varepsilon^0 \to 0$ in $L^2(\Omega,\mathbb{R}^3)$. On the other hand, by taking $V = \Phi$ in (2.9), where Φ is an arbitrary (but fixed in ε) vector field in $C_0^\infty(\Omega,\mathbb{R}^3)$ with $\operatorname{div}\Phi = 0$, we have, by integration by parts, that

$$\begin{aligned}\left|\int_\Omega H_{0,\mathrm{ex}}\cdot\Phi\right| = \left|\int_\Omega \frac{\operatorname{curl} B_\varepsilon^0\cdot \operatorname{curl}\Phi}{\rho_\varepsilon^2} + B_\varepsilon^0\cdot\Phi\right| &\leq C\left|\int_\Omega \operatorname{curl} B_\varepsilon^0\cdot\operatorname{curl}\Phi + B_\varepsilon^0\cdot\Phi\right| \\ &\leq C\left|\int_\Omega B_\varepsilon^0\cdot\operatorname{curl}\operatorname{curl}\Phi + B_\varepsilon^0\cdot\Phi\right| \\ &\leq C\left\|B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)}\|\Phi\|_{H^2(\Omega,\mathbb{R}^3)},\end{aligned}$$

where we used an integration by parts. Note that, as $\varepsilon\to 0$, the RHS converges to zero, while the LHS remains unchanged. This implies that

$$\int_\Omega H_{0,\mathrm{ex}}\cdot\Phi = 0, \quad \text{for any } \Phi\in C_0^\infty(\Omega,\mathbb{R}^3) \text{ such that } \operatorname{div}\Phi = 0.$$

Therefore, using (2.3), we conclude that $\int_\Omega H_{0,\mathrm{ex}}\cdot A = 0$ for any $A\in L^2_\sigma(\Omega,\mathbb{R}^3)$, which, by orthogonality, implies that $H_{0,\mathrm{ex}}$ can be written as $\nabla\zeta$ for some $\zeta\in H^1(\Omega)$.This means that $\operatorname{curl} H_{0,\mathrm{ex}} \equiv 0$ in Ω. □

4. Proof of the Main Result

Proof of Theorem 1.2. We divide the proof into four steps. The first three focus on establishing items (a) and (b), while the final step addresses the analysis beyond the first critical field.

Step 1. Proving that $F_{\varepsilon,\rho_\varepsilon}(u,A) = O(h_{\rm ex}^2)$. Since we are assuming that $(\mathbf{u},\mathbf{A})$ minimizes GL_ε, we have

$$GL_\varepsilon(\mathbf{u},\mathbf{A}) \leq GL_\varepsilon(\rho_\varepsilon e^{ih_{\rm ex}\phi_\varepsilon^0}, h_{\rm ex}A_\varepsilon^0). \tag{4.1}$$

This implies that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A|^2 \overset{(1.11)}{\leq} h_{\rm ex}\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 + \mathfrak{r}. \tag{4.2}$$

We note that, by integration by parts,

$$\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 = \int_\Omega \langle iu\,, \nabla_A u\rangle \cdot \operatorname{curl} B_\varepsilon^0 + \int_\Omega \operatorname{curl} A\cdot B_\varepsilon^0$$

Therefore, using Hölder's inequality, we are led to

$$\begin{aligned}
F_{\varepsilon,\rho_\varepsilon}(u,A) &\leq h_{\rm ex}\int_\Omega (\langle iu\,, \nabla_A u\rangle \cdot \operatorname{curl} B_\varepsilon^0 + \operatorname{curl} A\cdot B_\varepsilon^0) + \mathfrak{r} \\
&\leq C\Bigg(h_{\rm ex}\|u\|_{L^4(\Omega,\mathbb{C})}\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)}\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^4(\Omega,\mathbb{R}^3)} \\
&\qquad\qquad + \|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)}\left\|B_\varepsilon^0\right\|_{L^2(\Omega,\mathbb{R}^3)} + |\mathfrak{r}|\Bigg) \\
&\overset{(2.15)}{\leq} C(h_{\rm ex}\|u\|_{L^4(\Omega,\mathbb{C})}\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)} + \|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)} + |\mathfrak{r}|).
\end{aligned} \tag{4.3}$$

Let us now individually bound from above the terms on the RHS. For $\|u\|_{L^4(\Omega,\mathbb{C})}$, we use convexity to obtain

$$\|u\|_{L^4(\Omega,\mathbb{C})}^4 = \int_\Omega (1+(|u|^2-1))^2 \leq 2\int_\Omega 1 + (1-|u|^2)^2 \leq C(1+\varepsilon^2 F_{\varepsilon,\rho_\varepsilon}(u,A)),$$

and therefore

$$\|u\|_{L^4(\Omega,\mathbb{C})} \leq C(1+\varepsilon^{\frac12}F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac14}). \tag{4.4}$$

In the case of $\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)}$ and $\|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)}$, we have

$$\|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)} + \|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)} \leq CF_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac12}. \tag{4.5}$$

Finally, for $|\mathfrak{r}|$, we have that

$$\begin{aligned}
|\mathfrak{r}| &\leq \frac{h_{\rm ex}^2}{2}\int_\Omega \frac{|\operatorname{curl} B_\varepsilon^0|^2}{\rho_\varepsilon^2}|(|u|^2-1)| \\
&\overset{(1.6)}{\leq} Ch_{\rm ex}^2\left\|\operatorname{curl} B_\varepsilon^0\right\|_{L^4(\Omega,\mathbb{R}^3)}^2\left\|1-|u|^2\right\|_{L^2(\Omega,\mathbb{C})} \\
&\overset{(2.15)}{\leq} Ch_{\rm ex}^2\varepsilon F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac12}.
\end{aligned} \tag{4.6}$$

Thus, from (4.3), (4.4), (4.5), and (4.6), we conclude that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) = O\left(h_{\rm ex} F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac12}\right),$$

which implies that there exists $C>0$ such that, for any ε sufficiently small, we have

$$F_{\varepsilon,\rho_\varepsilon}(u,A) \le C h_{\rm ex}^2. \tag{4.7}$$

Step 2. Vorticity estimate. Proof of item (a). Since $h_{\rm ex} = O(|\log\varepsilon|)$, using (4.7), we have

$$F_{\varepsilon,\rho_\varepsilon}(u,A) \le C|\log\varepsilon|^2 \tag{4.8}$$

for some $C>0$ not depending on ε. This estimate allows for applying Theorem 1.1, which is crucial in deriving our quantitative estimates on $F_{\varepsilon,\rho_\varepsilon}(u,A)$ and $\mu(u,A)$. Using (1.14), we deduce that, for any $\gamma\in(0,1)$,

$$\begin{aligned}\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 &\le \|\mu(u,A)-\nu_\varepsilon\|_{(C_T^{0,\gamma}(\Omega,\mathbb{R}^3))^*}\left\|B_\varepsilon^0\right\|_{C_T^{0,\gamma}(\Omega,\mathbb{R}^3)} + \langle \nu_\varepsilon\,,B_\varepsilon^0\rangle\\ &\overset{(2.15)\&(4.8)}{\le} \langle \nu_\varepsilon\,,B_\varepsilon^0\rangle + C(|\log\varepsilon|^{2-\gamma q}).\end{aligned} \tag{4.9}$$

By inserting (1.13) and (4.9) into (4.2), and using (4.6) to control $|\mathfrak{r}|$, we deduce, by choosing a large enough n in (1.13), and hence a large enough q, that

$$\begin{aligned}F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\rm ex}\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 \ge \frac12|\rho_\varepsilon^2\nu_\varepsilon|\left(|\log\varepsilon| - C\log|\log\varepsilon|\right)\\ - h_{\rm ex}\langle\nu_\varepsilon\,,B_\varepsilon^0\rangle + o(|\log\varepsilon|^{-2}).\end{aligned} \tag{4.10}$$

On the other hand, by definition of R_ε (recall (1.15)), we have

$$\langle\nu_\varepsilon\,,B_\varepsilon^0\rangle \le |\rho_\varepsilon^2\nu_\varepsilon|\mathrm{R}_\varepsilon.$$

Inserting this into (4.10), we find that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\rm ex}\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 \ge \frac12|\rho_\varepsilon^2\nu_\varepsilon|\left(|\log\varepsilon| - 2h_{\rm ex}\mathrm{R}_\varepsilon - C\log|\log\varepsilon|\right) + o(|\log\varepsilon|^{-2}).$$

Using that $h_{\rm ex}\le H_{c_1}^\varepsilon - K_0\log|\log\varepsilon|$, we observe that the leading-order term in the RHS cancels out, allowing us to deduce

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\rm ex}\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 \ge \frac12|\rho_\varepsilon^2\nu_\varepsilon|(2K_0\mathrm{R}_\varepsilon - C)\log|\log\varepsilon| + o(|\log\varepsilon|^{-2}).$$

Observe that the assumption $\liminf_{\varepsilon\to0^+}\mathrm{R}_\varepsilon>0$ then allows us to choose K_0, independent of ε, such that $2K_0\mathrm{R}_\varepsilon - C>1$, leading us to

$$F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\rm ex}\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 \ge \frac12|\rho_\varepsilon^2\nu_\varepsilon|\log|\log\varepsilon| + o(|\log\varepsilon|^{-2}). \tag{4.11}$$

On the other hand, from (4.2) and (4.6), we deduce that

$$F_{\varepsilon,\rho_\varepsilon}(u,A) + \frac12\int_{\mathbb{R}^3\setminus\Omega}|\operatorname{curl}A|^2 - h_{\rm ex}\int_\Omega \mu(u,A)\cdot B_\varepsilon^0 \le |\mathfrak{r}| = O(\varepsilon|\log\varepsilon|^3). \tag{4.12}$$

Hence, by combining (4.11) with (4.12), we find that

$$|\rho_\varepsilon^2\nu_\varepsilon| = o(|\log\varepsilon|^{-1}).$$

Moreover, using (1.6) we find

$$|\nu_\varepsilon| \leq \frac{1}{b}|\rho_\varepsilon^2 \nu_\varepsilon| = o(|\log\varepsilon|^{-1}),$$

and therefore $\|\nu_\varepsilon\|_{(C_T^{0,\gamma}(\Omega,\mathbb{R}^3))^*} = o(|\log\varepsilon|^{-1})$ for any $\gamma \in (0,1]$. From the vorticity estimate (1.14), we then deduce that

$$\|\mu(u,A)\|_{(C_T^{0,\gamma}(\Omega,\mathbb{R}^3))^*} = o(|\log\varepsilon|^{-1}) \quad \text{for any } \gamma \in (0,1], \tag{4.13}$$

which finishes the proof of item (a).

Step 3. Meissner configuration approximation. Proof of item (b). Using (4.13), from (4.12) we find

$$F_{\varepsilon,\rho_\varepsilon}(u,A) + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A|^2 = o(1). \tag{4.14}$$

Finally, by inserting (4.14), (4.13), and (4.6) into the energy splitting (1.11), we conclude that

$$GL_\varepsilon(\mathbf{u},\mathbf{A}) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\text{ex}}\phi_\varepsilon^0}, h_{\text{ex}}A_\varepsilon^0) + o(1), \tag{4.15}$$

which means that item (b) is satisfied.

Step 4. Energetically optimal vortexless configuration beyond $H_{c_1}^\varepsilon$. Now suppose we have a configuration $(\mathbf{u},\mathbf{A})$ such that $GL_\varepsilon(\mathbf{u},\mathbf{A}) \leq GL_\varepsilon(\rho_\varepsilon e^{ih_{\text{ex}}\phi_\varepsilon^0}, h_{\text{ex}}A_\varepsilon^0)$, $|\mathbf{u}| > c > 0$ and $h_{\text{ex}} \leq \varepsilon^{-\alpha}$ for some $\alpha < \frac{1}{2}$. Note that we also have $|u| > c > 0$. By appealing to a vorticity estimate for configurations satisfying such a lower bound, as the one provided in [Rom19a, Proposition A.1], we obtain that there exists a constant $C = C(\Omega) > 0$ such that

$$\|\mu(u,A)\|_{(C_T^{0,1}(\Omega,\mathbb{R}^3))^*} \leq C\varepsilon F_\varepsilon(u,A),$$

where $F_\varepsilon(u,A)$ is the homogeneous free energy functional (that is, $F_{\varepsilon,\rho_\varepsilon}(u,A)$ with $\rho_\varepsilon \equiv 1$). In addition, from [Rom19b, Lemma 8.1], we have

$$\|\mu(u,A)\|_{(C^0(\Omega,\mathbb{R}^3))^*} \leq CF_\varepsilon(u,A).$$

Given $\gamma \in (0,1)$, we can then derive an estimate for $\|\mu(u,A)\|_{(C_T^{0,\gamma}(\Omega,\mathbb{R}^3))^*}$ by interpolating between the norms in $(C_T^{0,1}(\Omega,\mathbb{R}^3))^*$ and $(C^0(\Omega,\mathbb{R}^3))^*$. In fact, we have

$$\|\mu(u,A)\|_{(C_T^{0,\gamma}(\Omega,\mathbb{R}^3)^*} \leq \|\mu(u,A)\|^{\gamma}_{(C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \|\mu(u,A)\|^{1-\gamma}_{(C^0(\Omega,\mathbb{R}^3))^*} = C\varepsilon^\gamma F_\varepsilon(u,A).$$

Using the uniform bounds for ρ_ε (1.6), we conclude that

$$\|\mu(u,A)\|_{(C_T^{0,\gamma}(\Omega,\mathbb{R}^3)^*} \leq C\varepsilon^\gamma F_{\varepsilon,\rho_\varepsilon}(u,A), \tag{4.16}$$

where $C = C(\Omega,b) > 0$.

Since we assume that (4.1) holds, (4.2) remains valid. Then

$$\begin{aligned} F_{\varepsilon,\rho_\varepsilon}(u,A) + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A|^2 &\overset{(4.16)\&(2.16)\&(4.6)}{\leq} C\left(h_{\text{ex}}\varepsilon^\gamma F_{\varepsilon,\rho_\varepsilon}(u,A) + h_{\text{ex}}^2 \varepsilon F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac{1}{2}}\right) \\ &\overset{(4.7)}{\leq} C\varepsilon^\gamma h_{\text{ex}}^2 F_{\varepsilon,\rho_\varepsilon}(u,A)^{\frac{1}{2}}. \end{aligned}$$

It follows that

$$\left(F_{\varepsilon,\rho_\varepsilon}(u,A)+\frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega}|\operatorname{curl}A|^2\right)^{\frac{1}{2}}\leq C\varepsilon^{\gamma}h_{\mathrm{ex}}^2.$$

Since we are assuming $h_{\mathrm{ex}}\leq\varepsilon^{-\alpha}$ for $\alpha<\frac{1}{2}$, we choose $\gamma\in(2\alpha,1)$ to obtain

$$F_{\varepsilon,\rho_\varepsilon}(u,A)+\frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega}|\operatorname{curl}A|^2=o(1). \tag{4.17}$$

Inserting (4.16), (4.17), and (4.6) into the energy splitting (1.11), yields that (4.15) is valid when $h_{\mathrm{ex}}\leq\varepsilon^{-\alpha}$ for $\alpha<\frac{1}{2}$.

The proof is thus finished. □

5. ε-level estimates for a weighted Ginzburg–Landau functional

In this section, we provide the changes needed to adapt the proof of the ε-level estimates for the homogeneous Ginzburg–Landau functional in [Rom19b, Theorem 1.1], to the weighted case. The role played by the standard free-energy within Ω, that is,

$$F_\varepsilon(u,A)=\frac{1}{2}\int_\Omega|\nabla_A u|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2+|\operatorname{curl}A|^2,$$

will be replaced by $F_{\varepsilon,\rho_\varepsilon}(u,A)$. A crucial observation for adapting the arguments is that, since $0<c\leq\rho_\varepsilon\leq1$, we have that

$$c^4F_\varepsilon(u,A)\leq F_{\varepsilon,\rho_\varepsilon}(u,A)\leq F_\varepsilon(u,A). \tag{5.1}$$

In what follows, we use the notation

$$e_{\varepsilon,\rho_\varepsilon}(u):=\frac{\rho_\varepsilon^2}{2}|\nabla u|^2+\frac{\rho_\varepsilon^4}{4\varepsilon^2}(1-|u|^2)^2\quad\text{and}\quad e_{\varepsilon,\rho_\varepsilon}(u,A):=e_{\varepsilon,\rho_\varepsilon}(u)+|\operatorname{curl}A|^2.$$

The construction in [Rom19b] is lengthy and technical. For the reader's convenience, we will outline the key elements of the construction, though we will focus primarily on highlighting the main changes in the proofs. Interested readers are encouraged to track these changes alongside [Rom19b].

5.1. **Choice of grid.** Let us fix an orthonormal basis (e_1,e_2,e_3) of $\mathbb{R}^3$ and consider a grid $\mathfrak{G}=\mathfrak{G}(a,\delta)$ given by the collection of closed cubes $\mathscr{C}_i\subset\mathbb{R}^3$ of side-length $\delta=\delta(\varepsilon)$. In the grid we use a system of coordinates with origin in $a\in\Omega$ and orthonormal directions (e_1,e_2,e_3). From now on we denote by $\mathfrak{R}_1$ (respectively $\mathfrak{R}_2$) the union of all edges (respectively faces) of the cubes of the grid. We have the following lemma, which directly follows from [Rom19b, Lemma 2.1] by replacing $F_\varepsilon(u,A)$ by $F_{\varepsilon,\rho_\varepsilon}(u,A)$ in view of (5.1).

Lemma 5.1 (Choice of grid)**.** *For any $\gamma\in(-1,1)$ there exist constants $c_0(\gamma),c_1(\gamma)>0$, $\delta_0(\Omega)\in(0,1)$ such that, for any $\varepsilon,\delta>0$ satisfying*

$$\varepsilon^{\frac{1-\gamma}{2}}\leq c_0\quad\text{and}\quad c_1\varepsilon^{\frac{1-\gamma}{4}}\leq\delta\leq\delta_0,$$

if $(u,A)\in H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{R}^3)$ is a configuration such that $F_{\varepsilon,\rho_\varepsilon}(u,A)\leq\varepsilon^{-\gamma}$ then there exists $b_\varepsilon\in\Omega$ such that the grid $\mathfrak{G}(b_\varepsilon,\delta)$ satisfies

$$|u_\varepsilon|>5/8\quad\text{on }\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon,\delta))\cap\Omega, \tag{5.2a}$$

$$\int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon,\delta))\cap\Omega} E_{\varepsilon,\rho_\varepsilon}(u,A)d\mathcal{H}^1 \leq C\delta^{-2}F_{\varepsilon,\rho_\varepsilon}(u,A),$$

$$\int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon,\delta))\cap\Omega} E_{\varepsilon,\rho_\varepsilon}(u,A)d\mathcal{H}^2 \leq C\delta^{-1}F_{\varepsilon,\rho_\varepsilon}(u,A),$$

where C is a universal constant, and where hereafter $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure for $d \in \mathbb{N}$.

From now on we drop the cubes of the grid $\mathfrak{G}(b_\varepsilon,\delta)$ given by the previous lemma, whose intersection with $\partial\Omega$ is non-empty. We also define

$$\Theta := \Omega \setminus \cup_{\mathscr{C}_l\in\mathfrak{G}}\mathscr{C}_l \quad \text{and} \quad \partial\mathfrak{G} := \partial\left(\cup_{\mathscr{C}_l\in\mathfrak{G}}\mathscr{C}_l\right). \tag{5.3}$$

Observe that, in particular, $\partial\Theta = \partial\mathfrak{G} \cup \partial\Omega$.

We remark that $\mathfrak{G}(b_\varepsilon,\delta)$ carries a natural orientation. The boundary of every cube of the grid will be oriented accordingly to this orientation. Each time we refer to a face ω of a cube $\mathscr{C}$, it will be considered to be oriented with the same orientation of $\partial\mathscr{C}$. If we refer to a face $\omega \subset \partial\mathfrak{G}$, then the orientation used is the same of $\partial\mathfrak{G}$.

5.2. **Ball construction method on a surface for the weighted functional.** The primary modification required to adapt the ε-level estimates from [Rom19b] to the case of a weighted functional involves incorporating the weight when developing the ball construction method on a surface.

In this construction we will not need to obtain an estimate with ball per ball precision, but rather globally estimating the square of the weight by its minimum on the whole surface. Nevertheless, this is not trivial, since the weight appears with two different powers in the energy density $e_{\varepsilon,\rho_\varepsilon}(u)$ and, when bounding the energy from below, we need to extract the leading order power (i.e. ρ_ε^2, since $0 < \rho_\varepsilon \leq 1$).

Our main result in this section is the following modification of [Rom19b, Proposition 3.1]. It is worth recalling that this proposition extends the construction made by Sandier [San01] of the ball construction on a surface, when the number of vortices is bounded, and that follows the method of Jerrard [Jer99].

Proposition 5.1. *Let $\tilde{\Sigma}$ be a complete oriented surface in $\mathbb{R}^3$ whose second fundamental form is bounded by 1. Let Σ be a bounded open subset of $\tilde{\Sigma}$. For any $m, M > 0$ there exists $\varepsilon_0(m,M) > 0$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, if $u_\varepsilon \in H^1(\Sigma,\mathbb{C})$ satisfies*

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon,\Sigma) := \int_\Sigma e_{\varepsilon,\rho_\varepsilon}(u_\varepsilon) \leq M|\log\varepsilon|^m$$

and

$$|u_\varepsilon(x)| \geq \frac{1}{2} \quad \text{if } \mathfrak{d}(x,\partial\Sigma) < 1,$$

where $\mathfrak{d}(\cdot,\cdot)$ denotes the distance function in $\tilde{\Sigma}$, then, letting d be the winding number of $u_\varepsilon/|u_\varepsilon| : \partial\Sigma \to S^1$ and $\mathcal{M}_\varepsilon = M|\log\varepsilon|^m$ we have

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon,\Sigma) \geq \pi|d|\min_{\tilde{\Sigma}}\rho_\varepsilon^2\left(\log\frac{1}{\varepsilon} - \log\mathcal{M}_\varepsilon\right).$$

An immediate consequence of this result, is the following modification of [Rom19b, Corollary 3.1].

Corollary 5.1. *Let $\tilde{\Sigma}$ be a complete oriented surface in $\mathbb{R}^3$ whose second fundamental form is bounded by $Q_\varepsilon = Q|\log\varepsilon|^q$, where $q, Q > 0$ are given numbers. Let Σ be a bounded open subset of $\tilde{\Sigma}$. For any $m, M > 0$ there exists $\varepsilon_0(m, q, M, Q) > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, if $u_\varepsilon \in H^1(\Sigma, \mathbb{C})$ satisfies*

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, \Sigma) := \int_\Sigma e_{\varepsilon,\rho_\varepsilon}(u_\varepsilon) \leq M|\log\varepsilon|^m$$

and

$$|u_\varepsilon(x)| \geq \frac{1}{2} \quad \text{if } \mathfrak{d}(x, \partial\Sigma) < Q_\varepsilon^{-1},$$

where $\mathfrak{d}(\cdot,\cdot)$ denotes the distance function in $\tilde{\Sigma}$, then, letting d be the winding number of $u_\varepsilon/|u_\varepsilon| : \partial\Sigma \to S^1$ and $\mathcal{M}_\varepsilon = M|\log\varepsilon|^m$ we have

$$E_\varepsilon(u_\varepsilon, \Sigma) \geq \pi|d| \min_{\tilde{\Sigma}} \rho_\varepsilon^2 \left(\log\frac{1}{\varepsilon} - \log \mathcal{M}_\varepsilon Q_\varepsilon \right).$$

The proof of this result is a minor modification of the proof of [Rom19b, Corollary 3.1]. We include it here for the convenience of the reader.

Proof. For $y \in \Sigma_\varepsilon := Q_\varepsilon\Sigma$, we define

$$\tilde{u}_\varepsilon(y) = u_\varepsilon\left(\frac{y}{Q_\varepsilon}\right) \quad \text{and} \quad \tilde{\rho}_\varepsilon(y) = \rho_\varepsilon\left(\frac{y}{Q_\varepsilon}\right).$$

We let $\tilde{\Sigma}_\varepsilon := Q_\varepsilon\tilde{\Sigma}$. Observe that, by a change of variables, we have

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, \Sigma) = E_{\tilde{\varepsilon},\tilde{\rho}_\varepsilon}(\tilde{u}_\varepsilon, \Sigma_\varepsilon),$$

where $\tilde{\varepsilon} := \varepsilon Q_\varepsilon$. It is easy to check that the second fundamental form of $\tilde{\Sigma}_\varepsilon$ is bounded by 1. Then a direct application of Proposition 5.1 shows that

$$\begin{aligned} E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, \Sigma) = E_{\tilde{\varepsilon},\tilde{\rho}_\varepsilon}(\tilde{u}_\varepsilon, \Sigma_\varepsilon) \geq \pi|d| \min_{\tilde{\Sigma}_\varepsilon} \tilde{\rho}_\varepsilon^2 \left(\log\frac{1}{\tilde{\varepsilon}} - \log\mathcal{M}_\varepsilon \right) \\ = \pi|d| \min_{\tilde{\Sigma}} \rho_\varepsilon^2 \left(\log\frac{1}{\varepsilon} - \log\mathcal{M}_\varepsilon Q_\varepsilon \right) \end{aligned}$$

for any $0 < \varepsilon < \varepsilon_1 = \varepsilon_0 Q_\varepsilon^{-1}$, where ε_0 is the constant appearing in the proposition. □

Let us now focus on proving Proposition 5.1. In what follows, the sets $B(x,t)$ and $S(x,t)$ denote respectively a geodesic (closed) ball and a geodesic sphere of radius t centered at $x \in \tilde{\Sigma}$. We will also use $S_e(x,t)$ and $B_e(x,t)$ to denote, respectively, the Euclidean sphere and the Euclidean ball of radius t centered at x in the tangent space $T_x\Sigma$.

The following result is the main new ingredient of our ball construction. It is analogous to [Rom19b, Lemma 3.3] and [San01, Lemma 3.12] when $\rho_\varepsilon \equiv 1$.

Lemma 5.2. *Suppose u, Σ satisfy the hypotheses of Proposition 5.1. There exist $\varepsilon_0, r_0, C > 0$ depending only on Σ such that, for any $\varepsilon < \varepsilon_0$, any $x \in \Sigma$, and any $t > 0$ such that $\frac{\varepsilon}{\min_{B(x,t)} \rho_\varepsilon^2} < t < r_0$, if $|u| \leq 1$ in $S(x,t)$, we have*

$$E_{\varepsilon,\rho_\varepsilon}(u, S(x,t)) \geq \min_{\tilde{\Sigma}} \rho_\varepsilon^2 \left(\pi m^2 \left(\frac{|d|}{t} - C \right)^+ + \frac{1}{C\varepsilon}(1-m)^2 \right),$$

where $m := \inf_{S(x,t)} |u|$, and

$$d := \begin{cases} \deg\left(\frac{u}{|u|}, S(x,t)\right) & \text{if } m \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This result is an extension of [DVR24, Lemma A.1], which states that in $\Omega \subset \mathbb{R}^2$, for any $u \in H^1(\Omega)$ and any (Euclidean) ball $B_e(x,t) \subseteq \Omega$, we have

$$\frac{1}{2} \int_{S_e(x,t)} \rho_\varepsilon^2 |\nabla |u||^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{\min_{B_e(x,t)} \rho_\varepsilon^2}{C\varepsilon}(1-m)^2, \tag{5.4}$$

where $C > 0$ is a universal constant.

We extrapolate this result to surfaces in $\mathbb{R}^3$ using the exponential map defined locally in $\tilde{\Sigma}$. Fixing $u \in H^1(\Sigma, \mathbb{C})$, $x \in \Sigma$, $t > 0$ small enough, and using (5.4) with $\Omega = \exp_x^{-1}(\Sigma) \subseteq T_x\Sigma$, $\tilde{u} := u \circ \exp_x$, and $\tilde{\rho_\varepsilon} := \rho_\varepsilon \circ \exp_x$, yields

$$\frac{1}{2} \int_{S_e(x,t)} \tilde{\rho_\varepsilon}^2 |\nabla |\tilde{u}||^2 + \frac{\tilde{\rho_\varepsilon}^4}{2\varepsilon^2}(1-|\tilde{u}|^2)^2 \geq \frac{\min_{B_e(x,t)} \tilde{\rho_\varepsilon}^2}{C\varepsilon}(1-\tilde{m})^2, \tag{5.5}$$

where $\tilde{m} := \inf_{S(x,t)} |\tilde{u}|$. The bound on the second fundamental form of $\tilde{\Sigma}$ implies that there exists $r_0 > 0$ depending only on Σ such that, for $t < r_0$, $\exp_x$ is a quasi-isometry in $B_e(x,t)$ and $\exp_x(S_e(x,t))$ is a geodesic sphere in Σ centered at x. By performing a change of variables on (5.5), we obtain

$$\frac{1}{2} \int_{S(x,t)} \rho_\varepsilon^2 |\nabla |u||^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{\min_{B(x,t)} \rho_\varepsilon^2}{C\varepsilon}(1-m)^2 \geq \frac{\min_{\tilde{\Sigma}} \rho_\varepsilon^2}{C\varepsilon}(1-m)^2, \tag{5.6}$$

with potentially different values for the constants r_0 and C, after the change of coordinates. This yields the lower bound

$$\begin{aligned} E_{\varepsilon,\rho_\varepsilon}(u, S(x,t)) &= \frac{1}{2} \int_{S(x,t)} \rho_\varepsilon^2 \left(|\nabla |u||^2 + |u|^2 \left| \nabla \frac{u}{|u|} \right|^2 \right) + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2 \\ &\overset{(5.6)}{\geq} \min_{\tilde{\Sigma}} \rho_\varepsilon^2 \left(\frac{1}{C\varepsilon}(1-m)^2 + m^2 \int_{S(x,t)} \left| \nabla \frac{u}{|u|} \right|^2 \right). \end{aligned} \tag{5.7}$$

For the second term in the RHS of (5.7), we use the following result used in the proof of [San01, Lemma 3.12], which states that for small enough t,

$$\int_{S(x,t)} \left| \nabla \frac{u}{|u|} \right|^2 \geq \pi \left(\frac{|d|}{t} - C \right)^+. \tag{5.8}$$

By inserting (5.8) into (5.7), we complete the proof. ∎

With this modification at hand, the proof of Proposition 5.1 follows almost verbatim [Rom19b, Section 3.1], the only difference being that the lower bound $\min_{\tilde{\Sigma}} \rho_\varepsilon^2$ is carried through the subsequent steps.

5.3. **2D vorticity estimate.** Let ω be a two dimensional domain. For a given function $u : \overline{\omega} \to \mathbb{C}$ and a given vector field $A : \overline{\omega} \to \mathbb{R}^2$ we let

$$F_{\varepsilon,\rho_\varepsilon}(u, A, \omega) = \int_\omega e_{\varepsilon,\rho_\varepsilon}(u, A), \quad F_{\varepsilon,\rho_\varepsilon}(u, A, \partial\omega) = \int_{\partial\omega} e_{\varepsilon,\rho_\varepsilon}(u, A) d\mathcal{H}^1.$$

We also denote $F(u, A, \omega) = F_{\varepsilon,1}(u, A, \omega)$ and $F(u, A, \partial\omega) = F_{\varepsilon,1}(u, A, \partial\omega)$. Analog to (5.1), we have

$$c^4 F_\varepsilon(u, A, \omega) \le F_{\varepsilon,\rho_\varepsilon}(u, A, \omega) \le F_\varepsilon(u, A, \omega)$$

and

$$c^4 F_\varepsilon(u, A, \partial\omega) \le F_{\varepsilon,\rho_\varepsilon}(u, A, \partial\omega) \le F_\varepsilon(u, A, \partial\omega).$$

Hence, by combining these inequalities with [Rom19b, Theorem 4.1], we get the following 2D vorticity estimate.

Proposition 5.2. *Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Let $u : \omega \to \mathbb{C}$ and $A : \omega \to \mathbb{R}^2$ be $C^1(\overline{\omega})$ and such that $|u| \ge 5/8$ on $\partial\omega$. Let $\{U_j\}_{j\in J}$ be the collection of connected components of $\{|1 - |u(x)|| \ge 1/2\}$ and $\{S_i\}_{i\in I}$ denote the collection of connected components of $\{|u(x)| \le 1/2\}$ whose degree $d_i = \deg(u/|u|, \partial S_i) \ne 0$. Then, letting $r = \sum_{j\in J} \operatorname{diam}(U_j)$ and assuming $\varepsilon, r \le 1$, we have*

$$\left\| \mu(u, A) - 2\pi \sum_{i\in I} d_i \delta_{a_i} \right\|_{C^{0,1}(\omega)^*} \le C \max(\varepsilon, r)(1 + F_{\varepsilon,\rho_\varepsilon}(u, A, \omega) + F_{\varepsilon,\rho_\varepsilon}(u, A, \partial\omega)),$$

where a_i is the centroid of S_i and C is a universal constant.

Given a two dimensional Lipschitz domain $\omega \subset \Omega$, we let $(s, t, 0)$ denote coordinates in $\mathbb{R}^3$ such that $\omega \subset \{(s, t, 0) \in \Omega\}$. We define $\mu_\varepsilon := \mu_\varepsilon(u, A)[\partial_s, \partial_t]$, and write $\mu_{\varepsilon,\omega}$ its restriction to ω. From the previous proposition we immediately get the following result, which is analogous to [Rom19b, Corollary 4.1].

Corollary 5.2. *Let $\gamma \in (0, 1)$ and assume that $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_{\varepsilon,\rho_\varepsilon}(u, A) \le \varepsilon^{-\gamma}$, so that by Lemma 5.1 there exists a grid $\mathfrak{G}(b_\varepsilon, \delta)$ satisfying (5.2a). Then there exists $\varepsilon_0(\gamma)$ such that, for any $\varepsilon < \varepsilon_0$ and for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid $\mathfrak{G}(b_\varepsilon, \delta)$, letting $\{U_{j,\omega}\}_{j\in J_\omega}$ be the collection of connected components of $\{x \in \omega \mid |1 - |u(x)|| \ge 1/2\}$ and $\{S_{i,\omega}\}_{i\in I_\omega}$ denote the collection of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \le 1/2\}$ whose degree $d_{i,\omega} := \deg(u/|u|, \partial S_{i,\omega}) \ne 0$, we have*

$$\left\| \mu_{\varepsilon,\omega} - 2\pi \sum_{i\in I_\omega} d_{i,\omega} \delta_{a_{i,\omega}} \right\|_{C^{0,1}(\omega)^*} \le$$
$$C \max(r_\omega, \varepsilon) \left(1 + \int_\omega e_{\varepsilon,\rho_\varepsilon}(u, A) d\mathcal{H}^2 + \int_{\partial\omega} e_{\varepsilon,\rho_\varepsilon}(u, A) d\mathcal{H}^1 \right),$$

where $a_{i,\omega}$ is the centroid of $S_{i,\omega}$, $r_\omega := \sum_{j\in J_\omega} \operatorname{diam}(U_{j,\omega})$, and C is a universal constant.

5.4. **Construction of the vorticity approximation.** The construction of the vorticity approximation given in [Rom19b, Section 5] is exactly the same we need here. What changes when using $F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon)$ instead of $F(u_\varepsilon, A_\varepsilon)$ is the location of the set $\{|u_\varepsilon| \leq 1/2\}$, which is where the vortex lines are located. This is detected through the new version of the vorticity estimate given in Corollary 5.2. Even though our intention is not to repeat the whole construction here, let us next mention a few of the main ingredients for the convenience of the reader.

Let $\gamma \in (0,1)$ and a pair $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ such that $F_{\varepsilon,\rho_\varepsilon}(u, A) \leq \varepsilon^{-\gamma}$. We can then apply Lemma 5.1, which provides a grid of cubes $\mathfrak{G}(b_\varepsilon, \delta)$ satisfying (5.2). On the face ω of each cube $\mathscr{C}_l$ of the grid, Corollary 5.2 provides the existence of points $a_{i,\omega}$ and integers $d_{i,\omega} \neq 0$ such that

$$\mu_{\varepsilon,\omega} \approx 2\pi \sum_{i \in I_\omega} d_{i,\omega} \delta_{a_{i,\omega}}. \tag{5.9}$$

Our vorticity approximation in $\mathscr{C}_l$, which we denote $\nu_{\varepsilon,\mathscr{C}_l}$, is then constructed in such a way that its restriction to the face ω of any cube of the grid coincides with the right-hand side of (5.9). A crucial observation is that since $\partial\mu(u, A) = 0$ relative to any cube of the grid $\mathscr{C}_l$, we have

$$\sum_{\omega \subset \partial \mathscr{C}_l} \sum_{i \in I_\omega} d_{i,\omega} = 0,$$

which allows to define the vorticity approximation in any cube of the grid as the minimal connection, as first introduced in [BCL86], associated to the collection of positive and negative points (according to their degree) where the vortices are located on the boundary of the cube.

Analogously, since $\partial\mu(u, A) = 0$ relative to $\partial\mathfrak{G}$, we have

$$\sum_{\omega \subset \partial \mathfrak{G}} \sum_{i \in I_\omega} d_{i,\omega} = 0,$$

which allows to define the vorticity approximation in Θ (recall (5.3)), which we denote $\nu_{\varepsilon,\Theta}$, as the minimal connection through the boundary $\partial\Omega$ associated to the collection of positive and negative points where the vortices are located on $\partial\mathfrak{G}$.

It is important to remark that by using (5.1) we directly obtain a control of the size of the set where ν_ε is supported. In fact, by combining (5.1) with [Rom19b, Lemma 5.5], we obtain the following result.

Lemma 5.3. *Let $\gamma \in (0,1)$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$, so that, by Lemma 5.1, there exists a grid $\mathfrak{G}(b_\varepsilon, \delta)$ satisfying (5.2). For each face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid, let $|I_\omega|$ be the number of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$ whose degree is different from zero. Then, letting*

$$\mathfrak{G}_0 := \{\mathscr{C}_l \in \mathfrak{G} \mid \textstyle\sum_{\omega \subset \partial \mathscr{C}_l} |I_\omega| > 0\}, \tag{5.10}$$

we have

$$\operatorname{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} := \bigcup_{\mathscr{C}_l \in \mathfrak{G}_0} \mathscr{C}_l \cup \begin{cases} \overline{\Theta} & \text{if } \sum_{\omega \subset \partial \mathfrak{G}} |I_\omega| > 0 \\ \emptyset & \text{if } \sum_{\omega \subset \partial \mathfrak{G}} |I_\omega| = 0 \end{cases}.$$

Moreover

$$|S_{\nu_\varepsilon}| \leq C\delta(1 + \delta F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon)),$$

where C is a constant depending only on $\partial\Omega$.

5.5. **Lower bound for $E_\varepsilon(u_\varepsilon)$ far from the boundary.** We now proceed to provide a lower bound for the energy without magnetic field $E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon)$ in the union of cubes of the grid $\mathfrak{G}(b_\varepsilon,\delta)$. The proof is based on slicing according to the level sets of the smooth approximation of the function ζ constructed in [Rom19b, Appendix A]. The function ζ that we use here is exactly the same considered in [Rom19b]. It can be seen as a calibration of the minimal connection used when constructing the vorticity approximation. Its smooth approximation, coupled with quantitative estimates on the second fundamental form of its level sets, is one of the most technical parts of [Rom19b].

The aforementioned slicing procedure is coupled with the ball construction method on a surface of Section 5.2, where the quantitative estimate on the second fundamental form of the smooth approximation of the calibration function plays a crucial role.

The following result is analogous to [Rom19b, Proposition 6.1].

Proposition 5.3. *Let $m, M > 0$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^3)$ is such that $F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) = M_\varepsilon \le M|\log\varepsilon|^m$. For any $b, q > 0$, there exists $\varepsilon_0 > 0$ depending only on $b, q, m,$ and M, such that, for any $\varepsilon < \varepsilon_0$, letting $\mathfrak{G}(b_\varepsilon,\delta)$ denote the grid given by Lemma 5.1 with $\delta = \delta(\varepsilon) = |\log\varepsilon|^{-q}$, if —recall (5.10)—*

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, \cup_{\mathscr{C}_l\in\mathfrak{G}_0}\mathscr{C}_l) \le KM_\varepsilon \quad \text{and} \quad \sum_{\mathscr{C}_l\in\mathfrak{G}_0}\int_{\partial\mathscr{C}_l} e_{\varepsilon,\rho_\varepsilon}(u_\varepsilon)d\mathcal{H}^2 \le K\delta^{-1}M_\varepsilon$$

for some universal constant K, then

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, \cup_{\mathscr{C}_l\in\mathfrak{G}_0}\mathscr{C}_l) \ge \frac{1}{2}\sum_{\mathscr{C}_l\in\mathfrak{G}_0}\min_{\mathscr{C}_l}\rho_\varepsilon^2|\nu_{\varepsilon,\mathscr{C}_l}|\left(\log\frac{1}{\varepsilon} - \log C\frac{M_\varepsilon^{56}|\log\varepsilon|^{7(1+b)}}{\delta^{55}}\right) - \frac{C}{|\log\varepsilon|^b},$$

where C is a universal constant.

By replacing [Rom19b, Corollary 3.1] by Corollary 5.1, the proof of this result is a minor modification of the proof of [Rom19b, Proposition 6.1]. On each cube, one integrates over slices Σ of the smooth approximation of the calibration function, which makes appear the factor $\min_\Sigma \rho_\varepsilon^2$ on the lower bound. Since the level sets are contained in the cube, we have $\min_\Sigma \rho_\varepsilon^2 \ge \min_{\mathscr{C}_l}\rho_\varepsilon^2$, which can then be factored out, allowing the proof to continue exactly as in [Rom19b].

5.6. **Lower bound for $E_\varepsilon(u_\varepsilon)$ close to the boundary.** We now proceed to provide a lower bound for the energy without magnetic field $E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon)$ in Θ (recall (5.3)). The proof is based on slicing according to the level sets of the smooth approximation of the function ζ constructed in [Rom19b, Appendix B]. The function ζ that we use here is exactly the same considered in [Rom19b]. It can be seen as a calibration of the minimal connection through the boundary $\partial\Omega$ used when constructing the vorticity approximation. Once again, the slicing procedure is coupled with the ball construction method on a surface of Section 5.2, where the quantitative estimate on the second fundamental form of the smooth approximation of the calibration function plays a crucial role.

The following result is analogous to [Rom19b, Proposition 7.1].

Proposition 5.4. *Suppose that $\partial\Omega$ is of class C^2. Let $m, M > 0$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is such that $F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) = M_\varepsilon \leq M|\log\varepsilon|^m$. For any $b, q > 0$, there exists $\varepsilon_0 > 0$ depending only on b, q, m, M, and $\partial\Omega$, such that, for any $\varepsilon < \varepsilon_0$, letting $\mathfrak{G}(b_\varepsilon, \delta)$ denote the grid given by Lemma 5.1 with $\delta = \delta(\varepsilon) = |\log\varepsilon|^{-q}$, if*

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, \Theta) \leq KM_\varepsilon \quad \text{and} \quad \int_{\partial\mathfrak{G}} e_{\varepsilon,\rho_\varepsilon}(u_\varepsilon) d\mathcal{H}^2 \leq K\delta^{-1}M_\varepsilon$$

for some universal constant $K > 0$, then

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, \Theta) \geq \frac{1}{2} \min_{\overline{\Theta}} \rho_\varepsilon^2 |\nu_{\varepsilon,\Theta}| \left(\log\frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{124} |\log\varepsilon|^{(20+1/3)(1+b)}}{\delta^{123}} \right) - \frac{C}{|\log\varepsilon|^b},$$

where C is a constant depending only on $\partial\Omega$.

Once again, by replacing [Rom19b, Corollary 3.1] by Corollary 5.1, the proof of this result is a minor modification of the proof of [Rom19b, Proposition 7.1]. In Θ, one integrates over slices Σ of the smooth approximation of the calibration function, which makes appear the factor $\min_\Sigma \rho_\varepsilon^2$ on the lower bound. Since the level sets are contained in $\overline{\Theta}$, we have $\min_\Sigma \rho_\varepsilon^2 \geq \min_{\overline{\Theta}} \rho_\varepsilon^2$, which can then be factored out, allowing the proof to continue exactly as in [Rom19b].

5.7. **Proof of the ε-level estimates.** First, by replacing [Rom19b, Theorem 4.1] with Proposition 5.2, the proof of (1.14), that is, the vorticity estimate part of the ε-level estimates, is exactly the same as the one presented in [Rom19b, Section 8.1]. Next, we present the proof of the lower bound (1.13).

Proof. By gauge invariance of the energy $F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon)$, without loss of generality we can assume that A_ε is in the Coulomb gauge, that is,

$$\operatorname{div} A_\varepsilon = 0 \text{ in } \Omega \quad \text{and} \quad A_\varepsilon \cdot \nu = 0 \text{ on } \partial\Omega.$$

It directly follows that

$$\|A_\varepsilon\|_{H^1(\Omega,\mathbb{R}^3)} \leq C\|\operatorname{curl} A_\varepsilon\|_{L^2(\Omega,\mathbb{R}^3)},$$

where throughout the proof $C > 0$ denotes a constant that may change from line to line. By Sobolev embedding, we then have

$$\|A_\varepsilon\|_{L^p(\Omega,\mathbb{R}^3)} \leq C\|A_\varepsilon\|_{H^1(\Omega,\mathbb{R}^3)}$$

for any $1 \leq p \leq 6$.

Observe that

$$\frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla u_\varepsilon|^2 \leq \int_\Omega \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \rho_\varepsilon^2 |u_\varepsilon|^2 |A_\varepsilon|^2 \leq \int_\Omega |\nabla_{A_\varepsilon} u_\varepsilon|^2 + |u_\varepsilon|^2 |A_\varepsilon|^2. \tag{5.11}$$

Using Hölder's inequality, we find

$$\begin{aligned}
\int_\Omega |u_\varepsilon|^2 |A_\varepsilon|^2 &\leq \int_\Omega |A_\varepsilon|^2 + \int_\Omega (|u_\varepsilon|^2 - 1)|A_\varepsilon|^2 \\
&\leq \|A_\varepsilon\|_{L^6(\Omega)}^2 |\Omega|^{\frac{2}{3}} + \||u_\varepsilon|^2 - 1\|_{L^2(\Omega)} \|A_\varepsilon\|_{L^6(\Omega)}^2 |\Omega|^{\frac{1}{6}} \\
&\leq C\|\operatorname{curl} A_\varepsilon\|_{L^2(\Omega)}^2 (1 + \varepsilon F_\varepsilon(u_\varepsilon, A_\varepsilon)^{\frac{1}{2}}) \\
&\leq CF_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon)(1 + \varepsilon F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon)^{\frac{1}{2}}).
\end{aligned}$$

Combining this with (5.11), we find

$$E_{\varepsilon,\rho_\varepsilon}(u_\varepsilon) \leq C F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon).$$

Now, we consider the grid $\mathfrak{G}(b_\varepsilon, \delta)$ given by Lemma 5.1. It is easy to see that, up to an adjustment of the constant appearing in the lemma, we can require our grid to additionally satisfy the inequalities

$$\int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon,\delta))} e_{\varepsilon,\rho_\varepsilon}(u_\varepsilon) d\mathcal{H}^1 \leq C\delta^{-2} F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon),$$
$$\int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon,\delta))} e_{\varepsilon,\rho_\varepsilon}(u_\varepsilon) d\mathcal{H}^2 \leq C\delta^{-1} F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon).$$

Let us now consider the vorticity approximation ν_ε defined in Section 5.4 together with the set S_{ν_ε} (recall Lemma 5.3). Observe that

$$\begin{aligned}\int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla u_\varepsilon|^2 &= \int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \rho_\varepsilon^2 |u_\varepsilon|^2 |A_\varepsilon|^2 + 2\rho_\varepsilon^2 \langle \nabla_{A_\varepsilon} u_\varepsilon, i u_\varepsilon A_\varepsilon\rangle \\ &\leq \int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \rho_\varepsilon^2 |u_\varepsilon|^2 |A_\varepsilon|^2 + 2\rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon||u_\varepsilon||A_\varepsilon|.\end{aligned}$$

Let $t_\varepsilon > 0$. Using that $2xy \leq t_\varepsilon x^2 + \frac{1}{t_\varepsilon} y^2$, we have that

$$\int_{S_{\nu_\varepsilon}} 2\rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon||u_\varepsilon||A_\varepsilon| \leq t_\varepsilon \int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{t_\varepsilon} \int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |u_\varepsilon|^2 |A_\varepsilon|^2.$$

We then deduce that (recall that $\rho_\varepsilon^2 \leq 1$)

$$\begin{aligned}\int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla u_\varepsilon|^2 &\leq (1+t_\varepsilon) \int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \left(1 + \frac{1}{t_\varepsilon}\right) \int_{S_{\nu_\varepsilon}} |u_\varepsilon|^2 |A_\varepsilon|^2 \\ &= (1+t_\varepsilon) \int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \left(1 + \frac{1}{t_\varepsilon}\right) \left(\int_{S_{\nu_\varepsilon}} (|u_\varepsilon|^2 - 1)|A_\varepsilon|^2 + |A_\varepsilon|^2\right).\end{aligned}$$

Using Hölder's inequality, we find

$$\int_{S_{\nu_\varepsilon}} (|u_\varepsilon|^2 - 1)|A_\varepsilon|^2 \leq \||u_\varepsilon|^2 - 1\|_{L^2(S_{\nu_\varepsilon})} |S_{\nu_\varepsilon}|^{\frac{1}{6}} \|A_\varepsilon\|^2_{L^6(S_{\nu_\varepsilon}, \mathbb{R}^3)}$$

and

$$\int_{S_{\nu_\varepsilon}} |A_\varepsilon|^2 \leq |S_{\nu_\varepsilon}|^{\frac{2}{3}} \|A_\varepsilon\|^2_{L^6(S_{\nu_\varepsilon}, \mathbb{R}^3)}.$$

Hence, recalling (5.1), we deduce that

$$\begin{aligned}\int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla u_\varepsilon|^2 \leq \int_{S_{\nu_\varepsilon}} &\rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + t_\varepsilon F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) \\ &+ C\left(1 + \frac{1}{t_\varepsilon}\right) F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) \left(\varepsilon F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon)^{\frac{1}{2}} |S_{\nu_\varepsilon}|^{\frac{1}{6}} + |S_{\nu_\varepsilon}|^{\frac{2}{3}}\right).\end{aligned}$$

We let $t_\varepsilon = |\log\varepsilon|^{-(m+n)}$ and choose $\delta = \delta(\varepsilon)$ such that $\delta \leq |\log\varepsilon|^{-3(m+n)}$ (the final choice of this parameter will come after). Note that, $t_\varepsilon F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) \leq C|\log\varepsilon|^{-n}$, $|S_{\nu_\varepsilon}| \leq C\delta =$

$C|\log\varepsilon|^{-3(m+n)}$, and

$$\left(1+\frac{1}{t_\varepsilon}\right)F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon,A_\varepsilon)\left(\varepsilon F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon,A_\varepsilon)^{\frac12}|S_{\nu_\varepsilon}|^{\frac16}+|S_{\nu_\varepsilon}|^{\frac23}\right)\le C|\log\varepsilon|^{-n}.$$

In particular,

$$\int_{S_{\nu_\varepsilon}}\rho_\varepsilon^2|\nabla u_\varepsilon|^2\le\int_{S_{\nu_\varepsilon}}\rho_\varepsilon^2|\nabla_{A_\varepsilon}u_\varepsilon|^2+C|\log\varepsilon|^{-n}. \tag{5.12}$$

We now apply Proposition 5.3 and Proposition 5.4 with $b=n$ and $q=3(m+n)$. This yields the existence of $\varepsilon_0>0$, depending only on m,n,M, and $\partial\Omega$, such that, for any $0<\varepsilon<\varepsilon_0$,

$$E_\varepsilon(u_\varepsilon,S_{\nu_\varepsilon})\ge\frac12\Lambda_\varepsilon(\nu_\varepsilon,\rho_\varepsilon)\left(\log\frac1\varepsilon-\log C\frac{M_\varepsilon^{124}|\log\varepsilon|^{(20+1/3)(1+n)}}{\delta^{123}}\right)-\frac{C}{|\log\varepsilon|^n},$$

where

$$\Lambda_\varepsilon(\nu_\varepsilon,\rho_\varepsilon)=\min_{\overline{\Theta}}\rho_\varepsilon^2|\nu_{\varepsilon,\Theta}|+\sum_{\mathscr{C}_l\in\mathfrak{G}_0}\min_{\mathscr{C}_l}\rho_\varepsilon^2|\nu_{\varepsilon,\mathscr{C}_l}|, \tag{5.13}$$

$M_\varepsilon=F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon,A_\varepsilon)$, and C is a constant depending only on $\partial\Omega$. By combining this with (5.12), we are led to

$$\begin{aligned}\int_{S_{\nu_\varepsilon}}\rho_\varepsilon^2|\nabla_{A_\varepsilon}u_\varepsilon|^2+\frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+|\operatorname{curl}A_\varepsilon|^2&\\ \ge\Lambda_\varepsilon(\nu_\varepsilon,\rho_\varepsilon)\left(\log\frac1\varepsilon-\log C\frac{M_\varepsilon^{124}|\log\varepsilon|^{(20+1/3)(1+n)}}{\delta^{123}}\right)-\frac{C}{|\log\varepsilon|^n}.&\end{aligned} \tag{5.14}$$

We have reached the point where the hypotheses on ρ_ε are needed. On the one hand, since there exist $\alpha\in(0,1)$, $N>0$, and $C_1>0$ (that do not depend on ε) such that

$$\|\rho_\varepsilon\|_{C^{0,\alpha}(\Omega)}\le C_1|\log\varepsilon|^N,$$

we deduce that, for any cube $\mathscr{C}_l\in\mathfrak{G}_0$ and any $x\in\mathscr{C}_l$, we have (recall that $\rho_\varepsilon\le1$)

$$\left|\rho_\varepsilon^2(x)-\min_{\mathscr{C}_l}\rho_\varepsilon^2\right|\le2\left|\rho_\varepsilon(x)-\min_{\mathscr{C}_l}\rho_\varepsilon\right|\le2C_1\delta^\alpha|\log\varepsilon|^N.$$

It then follows that

$$\left|\min_{\mathscr{C}_l}\rho_\varepsilon^2|\nu_{\varepsilon,\mathscr{C}_l}|-|\rho_\varepsilon^2\nu_{\varepsilon,\mathscr{C}_l}|\right|\le C\delta^\alpha|\log\varepsilon|^N.$$

By choosing $\delta=\delta(\varepsilon)$ such that $\delta\le|\log\varepsilon|^{-\frac{n+N+1}{\alpha}}$, we conclude that

$$\left|\min_{\mathscr{C}_l}\rho_\varepsilon^2|\nu_{\varepsilon,\mathscr{C}_l}|-|\rho_\varepsilon^2\nu_{\varepsilon,\mathscr{C}_l}|\right||\log\varepsilon|\le C|\log\varepsilon|^{-n}. \tag{5.15}$$

On the other hand, since there exist $C_2>0$ and $\kappa>0$ (that do not depend on ε) such that a_ε is constant in the set $\{x\in\Omega\mid\operatorname{dist}(x,\partial\Omega)\le C_2|\log\varepsilon|^{-\kappa})\}$, from Proposition A, we deduce that ρ_ε is almost constant in the same set (which we make precise below). Then, by choosing $\delta=\delta(\varepsilon)$ such that $\delta\le|\log\varepsilon|^{-(\kappa+1)}$, we have that

$$\Theta\subset\{x\in\Omega\mid\operatorname{dist}(x,\partial\Omega)\le k(\varepsilon)\},$$

and therefore Proposition A implies that, for any $x \in \Theta$, we have

$$\left|\rho_\varepsilon^2(x) - \min_{\overline{\Theta}} \rho_\varepsilon^2\right| \le Ce^{-\frac{c}{\varepsilon|\log\varepsilon|^\kappa}}.$$

Hence,

$$\left|\min_{\overline{\Theta}} \rho_\varepsilon^2 |\nu_{\varepsilon,\Theta}| - |\rho_\varepsilon^2 \nu_{\varepsilon,\Theta}|\right| |\log\varepsilon| \le C|\log\varepsilon|^{-n}. \tag{5.16}$$

Finally, by choosing

$$\delta = \delta(\varepsilon) = \min\left(|\log\varepsilon|^{-3(m+n)}, |\log\varepsilon|^{-\frac{n+N+1}{\alpha}}, |\log\varepsilon|^{-(\kappa+1)}\right),$$

by combining (5.14), (5.15), and (5.16), we find

$$\begin{aligned}\int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |\operatorname{curl} A_\varepsilon|^2 \\ \ge \left(|\rho_\varepsilon^2 \nu_{\varepsilon,\Theta}| + \sum_{\mathscr{C}_l \in \mathfrak{G}_0} |\rho_\varepsilon^2 \nu_{\varepsilon,\mathscr{C}_l}|\right)(|\log\varepsilon| - C\log|\log\varepsilon|) - \frac{C}{|\log\varepsilon|^n} \\ = |\rho_\varepsilon^2 \nu_\varepsilon| (|\log\varepsilon| - C\log|\log\varepsilon|) - \frac{C}{|\log\varepsilon|^n}.\end{aligned}$$

This conclude the proof of the lower bound. ☐

Remark 5.1. *We note that if the hypotheses on a_ε in Theorem 1.1 are not satisfied, then we obtain the lower bound*

$$\begin{aligned}\int_{S_{\nu_\varepsilon}} \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |\operatorname{curl} A_\varepsilon|^2 \\ \ge \Lambda_\varepsilon(\nu_\varepsilon, \rho_\varepsilon)(|\log\varepsilon| - C\log|\log\varepsilon|) - \frac{C}{|\log\varepsilon|^n},\end{aligned}$$

where $\Lambda_\varepsilon(\nu_\varepsilon, \rho_\varepsilon)$ is defined in (5.13). By replacing (1.13) with this inequality, and performing a similar analysis to that of the article, one obtains a lower bound for the first critical field in terms of a modified (and significantly more intricate) version of the weighted isoflux problem, which, in particular, depends on the grid used in the proof of the ε-level estimates.

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